Multiple operator integrals and the abstract pseudodifferential calculus of Connes and Moscovici

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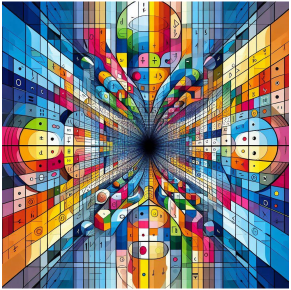

Based on ongoing joint work with Eva-Maria Hekkelman (UNSW) and Edward McDonald (Penn State)

Part I: Spotting multiple operator integrals in nature

Part II: The abstract pseudodifferential calculus of Connes and Moscovici

Part I

Spotting multiple operator integrals in nature

Part I.a:
MOIs in Connes' approach to particle physics

Noncommutative geometry

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a *-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and a self-adjoint operator D, acting in the Hilbert space $\mathcal{H}$, such that $(D - i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

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Algebra: $\mathcal{A} := C^\infty(S^1)$

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$[D, a]\psi = D(a\cdot\psi) - a\cdot D(\psi) = (-i)\frac{d}{d\theta}(a\cdot\psi) - (-i)a\frac{d}{d\theta}\psi = (-i)\frac{da}{d\theta}\psi$

$$\boxed{[D, a] = -i\frac{da}{d\theta}}$$

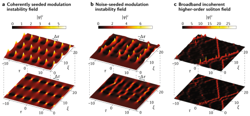
Connes' Reconstruction theorem: If $\mathcal{A}$ is commutative (and 8 technical properties hold) then $(\mathcal{A}, \mathcal{H}, D)$ must be of the form

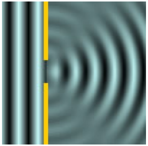
$$(C^\infty(M), L^2(E), D_M),$$

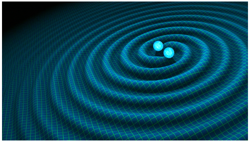
for a Riemannian manifold M, a spinor bundle $E \to M$, and D_M the Dirac operator in $L^2(E)$.

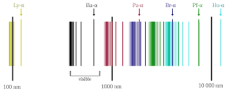
The spectral action:

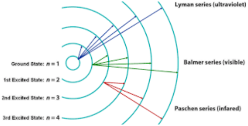
$\mathrm{Tr}(f(D))$, for a suitable function $f : \mathbb{R} \to \mathbb{R}$ and a self-adjoint operator D whose spectrum encodes 'the physics'.

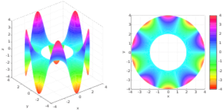


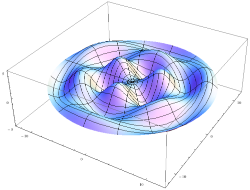












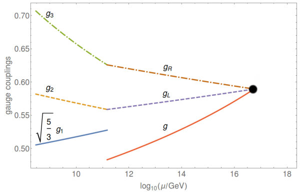
Any countably additive way to obtain a number from this spectrum is of the form $\mathrm{Tr}(f(D))$.

Two ways to expand the spectral action $\mathrm{Tr}(f(\frac{D+V}{\Lambda}))$:

1. Expand $\mathrm{Tr}(f(\frac{D+V}{\Lambda}))$ in Λ

$$\mathrm{Tr}(f(\frac{D_{M\times F}+V}{\Lambda})) = c_0\Lambda^4 \mathrm{vol}(M) + c_1\Lambda^2 \int R\sqrt{g}\,dx + c_2 \int \mathrm{tr} F_{\mu\nu}F^{\mu\nu} - c_3 \int |\phi|^2 + c_4 \int |\phi|^4 + \cdots$$

Spectral triple→Physical effective action, RG flow → measurable data. But: noncommutativity is ignored in intermediate step.



Renormalization Group flow

cf. [van Suijlekom, Chamseddine, Connes, JHEP]

2. Expand $\mathrm{Tr}(f(\frac{D+V}{\Lambda}))$ in $\Lambda^{-1}V$

Taylor: $\mathrm{Tr}(f(\frac{D+V}{\Lambda})) = \sum_{n=0}^{\infty} \frac{\Lambda^{-n}}{n!} \mathrm{Tr}(T_{f^{[n]}}^{D/\Lambda,\ldots,D/\Lambda}(V,\ldots,V))$

Part I.b:
MOIs in the wild noncommutative literature

One may spot MOIs throughout the noncommutative literature.

e.g.

We use the Chern character of $(\mathcal{A}, \mathcal{H}, D)$ in entire cyclic cohomology (cf. [2]) given in the most efficient manner by the JLO formula, which defines the components of an entire cocycle in the (b, B) bicomplex:

$$\psi_n(a^0, \ldots, a^n) = \sqrt{2i} \int_{\sum_0^n v_i = 1, v_i \geq 0} \tag{90}$$

$$\operatorname{Trace}\left(a^0\, e^{-v_0 D^2}\, [D, a^1]\, e^{-v_1 D^2} \ldots e^{-v_{n-1} D^2}\, [D, a^n]\, e^{-v_n D^2}\right), \qquad \forall a^j \in \mathcal{A}$$

where n is odd.

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and

1.1. **Rearrangement Lemma and multivariable functional calculus.** An important technical tool for the calculation of heat coefficients in the noncommutative setting is the Rearrangement Lemma which informally reads

$$\int_0^\infty f_0(uk^2) \cdot b_1 \cdot f_1(uk^2) \cdot b_2 \cdot \ldots \cdot b_p \cdot f_p(uk^2)\, du$$
$$= k^{-2} F(\Delta^{(1)}, \Delta^{(1)}\Delta^{(2)}, \dots, \Delta^{(1)} \cdot \ldots \cdot \Delta^{(p)})(b_1 \cdot \ldots \cdot b_p), \tag{1.4}$$

where the function $F(s_1, \dots, s_p)$ is

$$F(s) = \int_0^\infty f_0(u) \cdot f_1(us_1) \cdot \ldots \cdot f_p(us_p)\, du$$

and $\Delta^{(j)}$ signifies that the modular operator $\Delta = k^{-2} \cdot k^2$ acts on the j-th factor. In [CoMo14] it is proved for the concrete integral

$$\int_0^\infty (uk^2)^{|\alpha|+p-1}(1+uk^2)^{-\alpha_0-1} \cdot b_1 \cdot (1+uk^2)^{-\alpha_1-1} \cdot \ldots \cdot b_p \cdot (1+uk^2)^{-\alpha_p-1}\, du, \tag{1.5}$$

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The computation of this matrix-valued function $\mathscr{K}$ is based on the Volterra series

$$e^{A+B} = e^A + \sum_{k=1}^\infty \int_{\Delta_k} ds\, e^{(1-s_1)A} B\, e^{(s_1-s_2)A} \cdots e^{(s_{k-1}-s_k)A} B\, e^{s_k A},$$

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Theorem 4.3 (Odd semifinite local index formula)**.** *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd finitely summable QC^∞ spectral triple with spectral dimension $q \geq 1$. Let $N = [q/2]+1$ where $[\cdot]$ denotes the integer part, and let $u \in \mathcal{A}$ be unitary. Then*

1) $$index(QuQ) = \frac{1}{\sqrt{2\pi i}} res_{r=(1-q)/2} \left(\sum_{m=1,odd}^{2N-1} \phi_m^r(Ch_m(u))\right)$$

where for $a_0, \ldots, a_m \in \mathcal{A}$, $l = \{a + iv : v \in \mathbf{R}\}$, $0 < a < 1/2$, $R_s(\lambda) = (\lambda - (1 + s^2 + \mathcal{D}^2))^{-1}$ and $r > 0$ we define $\phi_m^r(a_0, a_1, \ldots, a_m)$ to be

$$\frac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)} \int_0^\infty s^m \tau \left(\frac{1}{2\pi i} \int_l \lambda^{-q/2-r} a_0 R_s(\lambda)[\mathcal{D}, a_1] R_s(\lambda) \cdots [\mathcal{D}, a_m] R_s(\lambda) d\lambda\right) ds.$$

In particular the sum on the right hand side of 1) analytically continues to a deleted neighbourhood of $r = (1-q)/2$ with at worst *a simple pole at $r = (1-q)/2$. Moreover, the complex function-valued cochain $(\phi_m^r)_{m=1,odd}^{2N-1}$ is a (b, B) cocycle for $\mathcal{A}$ modulo functions holomorphic in a half-plane containing $r = (1-q)/2$.*

……

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$$\sum_{i_0,\ldots,i_n=1}^{m} f^{[n]}(\lambda_{i_0},\ldots,\lambda_{i_n})(V_1)_{i_0i_1}\cdots(V_n)_{i_{n-1}i_n}E_{i_0i_n}$$

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Spoiler alert: they are all the same.
$=: T^{D,\ldots,D}_{f^{[n]}}(V_1,\ldots,V_n)$.

Let $H_0, \dots, H_n$ be self-adjoint in $\mathcal{H}$.
Suppose $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is measurable and can be written as

$$\phi(\lambda_0, \dots, \lambda_n) = \int_\Sigma \alpha_0(\lambda_0, \sigma) \cdots \alpha_n(\lambda_n, \sigma) d\sigma$$

for a finite measure space (Σ, σ), and bounded measurable $\alpha_j : \Sigma \times \mathbb{R} \to \mathbb{R}$.
We define [Peller 2006] the multiple operator integral

$$T_\phi^{H_0,\dots,H_n}(V_1, \dots, V_n)\psi := \int_\Sigma \alpha_0(H_0, \sigma) V_1 \alpha_1(H_1, \sigma) \cdots V_n \alpha_n(H_n, \sigma)\psi \, d\sigma.$$

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This defines a well-defined multilinear operator

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One often sees $\phi = f^{[n]}$. If $\widehat{f^{(n)}} \in L^1$ then $\phi = f^{[n]}$ splits as above. (Because of .)
Credit $n = 1$: Daletskii, Krein, Löwner, Krein, Birman, Solomyak

For the analysts, MOIs are nice because they lead to sharp bounds:

- $\|f(D+V)-f(D)\|_p \leqslant c_p \|f\|_{\mathrm{Lip}} \|V\|_p$ [Potatov,Sukochev]
- $\|[f(D),V]\|_p \leqslant C_p \|f\|_{\mathrm{Lip}} \|[D,V]\|_p$, $C_p \sim \frac{p^2}{p-1}$ [Caspers,Montgomery-Smith,Potapov,Sukochev]
- $\|\frac{d^n}{dt^n} f(D+tV)|_{t=0}\|_p \leqslant c_p \|f^{(n)}\|_\infty \|V\|_p^n$[Potatov,Sukochev,Skripka]

where $p \in (1,\infty)$.
Direct applications are spectral shift functions, but also the sharpness of the above results is quite helpful.

In the Journal of Soviet Mathematics, 1993:

OPERATOR INTEGRATION, PERTURBATIONS, AND COMMUTATORS

M. Sh. Birman and M. Z. Solomyak

UDC 517.43

Under mild assumption, integral representations of the form

$$f(A_1)\cdot J - J\cdot f(A_0) = \iint \frac{f(\mu)-f(\lambda)}{\mu-\lambda}\, dE_1(\mu)(A_1 J - J A_0)\, dE_0(\mu), \qquad (*)$$

are justified. Here A_k, $k = 0, 1$, is a self-adjoint operator in a Hilbert space $\mathcal{H}_k$, J is an operator from $\mathcal{H}_0$ into $\mathcal{H}_1$; in general, all the operators are unbounded; E_k is the spectral measure of the operator A_k. On the basis of the representation (*), estimates of the s-numbers of the operator $f(A_1)\cdot J - J\cdot f(A_0)$ in terms of the s-numbers of the operator $A_1 J - J A_0$ are given. Analogous results are obtained for commutators and anticommutators.

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Under mild assumption, integral representations of the form

$$f(A_1)\cdot \mathcal{J} - \mathcal{J}\cdot f(A_0) = \iint \frac{f(\mu)-f(\lambda)}{\mu-\lambda}\, dE_1(\mu)(A_1\mathcal{J}-\mathcal{J}A_0)\,dE_0(\mu)\,, \qquad (*)$$

are justified. Here A_k, $k = 0, 1$, is a self-adjoint operator in a Hilbert space $\mathcal{H}_k$, $\mathcal{J}$ is an operator from $\mathcal{H}_0$ into $\mathcal{H}_1$; in general, all the operators are unbounded; E_k is the spectral measure of the operator A_k. On the basis of the representation (*), estimates of the s-numbers of the operator $f(A_1)\cdot\mathcal{J} - \mathcal{J}\cdot f(A_0)$ in terms of the s-numbers of the operator $A_1\mathcal{J} - \mathcal{J}A_0$ are given. Analogous results are obtained for commutators and anticommutators.

In MOI notation:

1. $f(A) - f(B) = T^{A,B}_{f^{[1]}}(A - B)$
2. $[f(H), a] = T^{H,H}_{f^{[1]}}([H, a])$

Here, $f^{[1]}(\mu, \lambda) = \frac{f(\mu)-f(\lambda)}{\mu-\lambda}$.

Adding the 0th order $f(H) = T^{H}_{f[0]}()$ one realises the two relations relate MOIs of 0th order to MOIs of 1st order.

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Generalise to higher order:

0. $f(H) = T^{H}_{f^{[0]}}()$
1. $T^{H_0,\dots,A,\dots,H_n}_{f^{[n]}}(V_1, \dots, V_n) - T^{H_0,\dots,B,\dots,H_n}_{f^{[n]}}(V_1, \dots, V_n) = T^{H_0,\dots,A,B,\dots,H_n}_{f^{[n+1]}}(V_1, \dots, A - B, \dots, V_n)$
2. $T^{H_0,\dots,H_n}_{f^{[n]}}(\dots, V_{j-1}, aV_j, \dots) - T^{H_0,\dots,H_n}_{f^{[n]}}(\dots, V_{j-1}a, V_j, \dots) = T^{H_0,\dots,H_n}_{f^{[n+1]}}(\dots, V_{j-1}, [H_j, a], V_j, \dots)$

Theorem ()

For all $N \in \mathbb{N}$: $f(H+V) = \sum_{n=0}^{N} T_{f[n]}^{H,\dots,H}(V,\dots,V) + T_{f[N+1]}^{H+V,H,\dots,H}(V,\dots,V)$

Proof.

Induction basis ($N = 0$):

$$\begin{aligned} f(H+V) &\overset{(1)}{=} f(H) + T_{f[1]}^{H+V,H}(V) \\ &\overset{(0)}{=} T_{f[0]}^{H}() + T_{f[1]}^{H+V,H}(V). \end{aligned}$$

Induction step:

$$\begin{aligned} f(H+V) &\overset{(IH)}{=} \sum_{n=0}^{N} T_{f[n]}^{H,\dots,H}(V,\dots,V) + T_{f[N+1]}^{H+V,H,\dots,H}(V,\dots,V) \\ &= \sum_{n=0}^{N+1} T_{f[n]}^{H,\dots,H}(V,\dots,V) + T_{f[N+1]}^{H+V,H,\dots,H}(V,\dots,V) \\ &\qquad - T_{f[N+1]}^{H,H,\dots,H}(V,\dots,V) \\ &\overset{(1)}{=} \sum_{n=0}^{N+1} T_{f[n]}^{H,\dots,H}(V,\dots,V) + T_{f[N+2]}^{H+V,H,\dots,H}(V,\dots,V) \end{aligned}$$

□

Theorem (humanity)

For all $N \in \mathbb{N}$: $f(H+V) = \sum_{n=0}^{N} T_{f[n]}^{H,\ldots,H}(V,\ldots,V) + T_{f[N+1]}^{H+V,H,\ldots,H}(V,\ldots,V)$

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$$= \sum_{n=0}^{N+1} T_{f[n]}^{H,\ldots,H}(V,\ldots,V) + T_{f[N+1]}^{H+V,H,\ldots,H}(V,\ldots,V) - T_{f[N+1]}^{H,H,\ldots,H}(V,\ldots,V)$$
$$\overset{(1)}{=} \sum_{n=0}^{N+1} T_{f[n]}^{H,\ldots,H}(V,\ldots,V) + T_{f[N+2]}^{H+V,H,\ldots,H}(V,\ldots,V)$$

□

Hence, it follows from just 0 and 1 that

$$\mathrm{Tr}(f(D+tV)) \sim \sum_{n=1}^{\infty} t^n \,\mathrm{Tr}(T^D_{f^{[n]}}(V,\dots,V)).$$

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Similarly, it follows from 1 and 2 and cyclicity that the functionals

$$\phi_n(a_0,\ldots,a_n) = \mathrm{Tr}(a_0[D,a_1]\,T^D_{f'^{[n]}}([D,a_1],\ldots,[D,a_n]))$$

are (b,B)-cocycles for even n.

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Adding some very general summability assumptions, one finds $\frac{d^n}{dt^n} f(D+tV)|_{t=0} = T^{D}_{f^{[n]}}(V,\dots,V)$, convergence of the Taylor series, and entire cyclic cocycles that recover the spectral action:
$\mathrm{Tr}(f(D+V)-f(D)) = \sum_{k=1}^{\infty}\left(c_k \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k}\int_{\phi_{2k}} F^k\right)$. [van Suijlekom-vN,2021]
Connes asked what happens to ϕ_n if $D \mapsto \frac{D}{\Lambda}$ and $\Lambda \to \infty$.. "not obvious at all"
Indeed, an answer requires unbounded multiple operator integrals!

Another reason for unbounded MOIs

From [vN-Sukochev-Zanin,2023]:

The local invariants $I_k(P)$ of an operator P acting in $L_2(\mathbb{T}^d_\theta)$ are the unique coefficients occurring in the heat trace expansion, which is the asymptotic expansion

$$\mathrm{Tr}(\lambda_l(y)e^{-tP}) \sim \sum_{\substack{k\geq 0\\ k=0\mathrm{mod}2}} t^{\frac{k-d}{2}}\tau(y\, I_k(P)), \quad t\downarrow 0 \qquad (y\in L_\infty(\mathbb{T}^d_\theta)). \tag{1.1}$$

In [42] it was shown that this expansion exists if (and in particular e^{-tP} is trace class if) P is self-adjoint and of the form

$$P = \lambda_l(x)\Delta + \sum_{i=1}^d \lambda_l(a_i)D_i + \lambda_l(a) \quad \text{for some} \quad x, a_i, a \in C^\infty(\mathbb{T}^d_\theta), \tag{1.2}$$

and later on:

$$\mathbf{W}_j^{\mathscr{A},\iota} = \begin{cases}\mathbf{A}_{\iota(j)} & (j\in\mathscr{A});\\ \mathbf{P}, & (j\notin\mathscr{A}),\end{cases} \tag{3.5}$$

where (for all $i\in\{1,\dots,d\}$)

$$\mathbf{A}_i := 2x\mathbf{D}_i + a_i, \quad \mathbf{P} := x\sum_{i=1}^d \mathbf{D}_i^2 + \sum_{i=1}^d a_i\mathbf{D}_i + a \quad \in \quad \mathcal{X}. \tag{3.6}$$

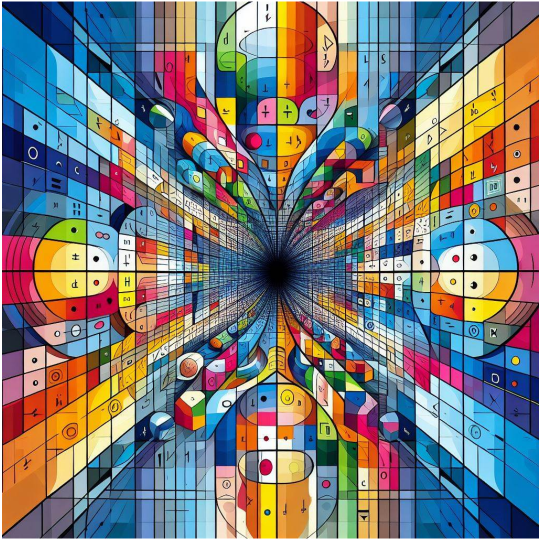
3.2. **Main result.** Our main result is formulated as follows.

Theorem 3.3. *Let $d\in\mathbb{N}_{\geq 2}$, $k\in 2\mathbb{Z}_+$, and let P be a self-adjoint operator acting in $L_2(\mathbb{T}^d_\theta)$ of the form (1.2) for positive invertible x. The k^{th} order local invariant of P occurring in the asymptotic expansion (1.1) takes the form*

(3.7)
$$I_k(P) = (-1)^{\frac{k}{2}}\pi^{\frac{d}{2}} \sum_{\frac{k}{2}\leq m\leq k}\ \sum_{\substack{\mathscr{A}\subseteq\{1,\dots,m\}\\ |\mathscr{A}|=2m-k}}\ \sum_{\iota:\mathscr{A}\to\{1,\dots,d\}} c_d^{(\iota)}\mathbf{T}^{x,m}_{F_{k,d}}(\mathbf{W}_1^{\mathscr{A},\iota},\dots,\mathbf{W}_m^{\mathscr{A},\iota}),$$

Part II

The abstract Ψ differential calculus of Connes and Moscovici



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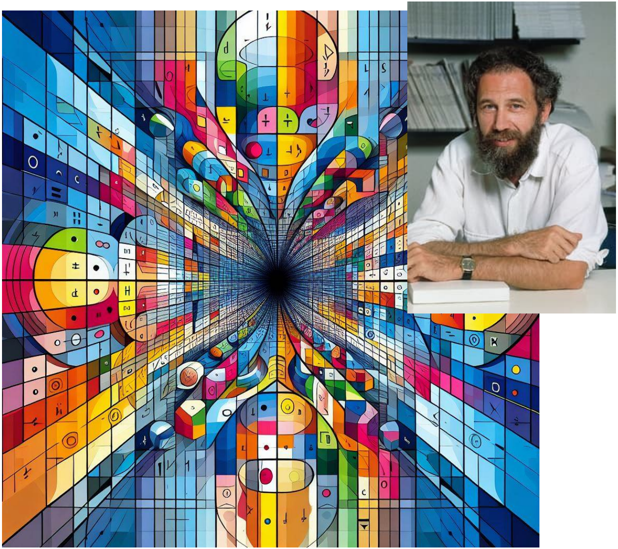

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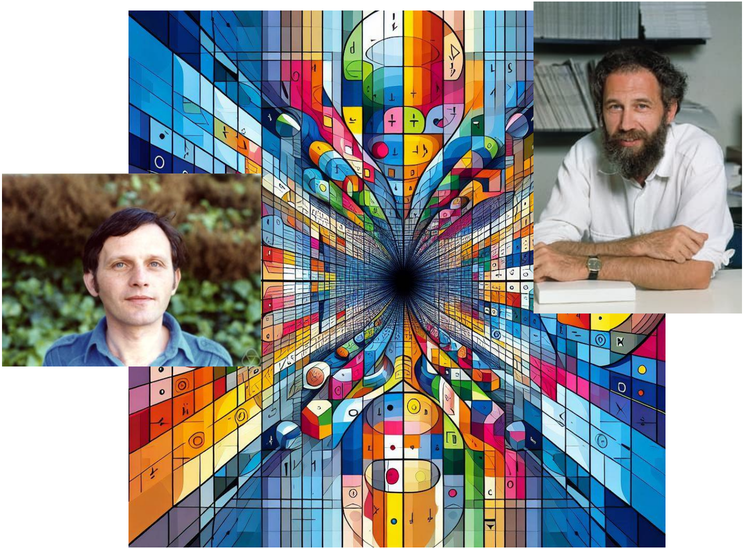

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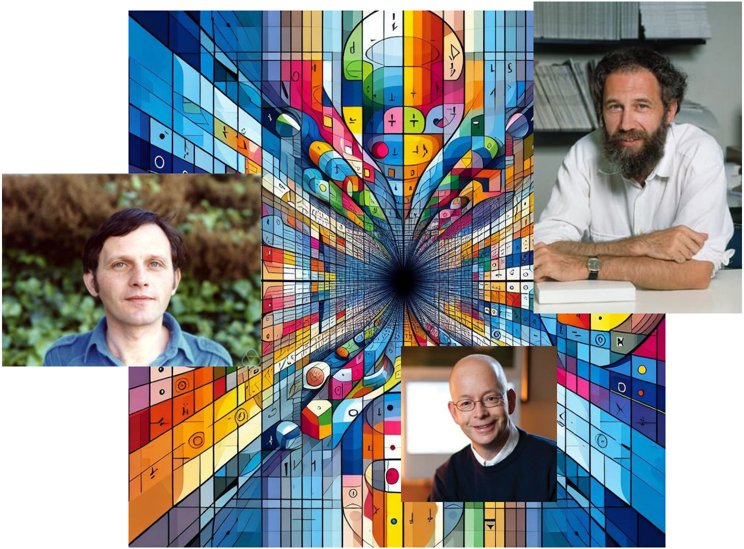

Part II

The abstract Ψ differential calculus of Connes and Moscovici

The pseudodifferential calculus:

Let Θ be a positive invertible operator in $\mathcal{H}$. (Think of $\sqrt{1+D^2}$.)

- Define the Hilbert spaces

$$\mathcal{H}^s := \overline{\mathrm{dom}\Theta^s}^{\|\cdot\|_s}, \qquad \langle \phi, \psi \rangle_{\mathcal{H}^s} := \langle \Theta^s \phi, \Theta^s \psi \rangle$$

for $s \in \mathbb{R}$ where $\|\phi\|_{\mathcal{H}^s} := \|\Theta^s \phi\|$ – though taking this closure is not necessary for $s \geqslant 0$. We write $\mathcal{H}^\infty = \bigcap_{s \geqslant 0} \mathcal{H}^s$, which is dense in $\mathcal{H}$.

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- We say that a linear operator $A : \mathcal{H}^\infty \to \mathcal{H}^\infty$ is in the class

$$\mathrm{op}^r = \mathrm{op}^r(\Theta)$$

if A extends to a continuous operator

$$\overline{A}^{s,r} : \mathcal{H}^{s+r} \to \mathcal{H}^s$$

for all $s \in \mathbb{R}$. (If no confusion can arise, we write $A : \mathcal{H}^{s+r} \to \mathcal{H}^s$.)

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- We define $\Psi^r \subseteq \mathrm{op}^r$ as those $A \in \mathrm{op}^r$ for which $\delta_\Theta^n(A) \in \mathrm{op}^r$ for each $n \geqslant 0$, where $\delta_\Theta(A) := [\Theta, A]$.

Theorem

Let $n \in \mathbb{N}$, let $H_0, \dots H_n$ be self-adjoint operators in $\mathcal{H}$, and let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be of the form

$$\phi(\lambda_0, \dots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega),$$

for a finite measure space (Ω, ν) and bounded measurable $a_j : \mathbb{R} \times \Omega \to \mathbb{C}$. Suppose that we have $a_j(H_j, \omega) \in \mathrm{op}^0(\Theta)$ and

$$\|a_j(H_j, \omega)\|_{\mathcal{H}^s \to \mathcal{H}^s} \leqslant C_{s,H_j} \|a_j(\cdot, \omega)\|_\infty$$

for every $0 \leqslant j \leqslant n$, $s \in \mathbb{R}$, and $\omega \in \Omega$, and certain constants $C_{s,H_j} \in \mathbb{R}$. Then the integral

$$T_\phi^{H_0,\dots,H_n}(V_1, \dots, V_n)\psi := \int_\Omega a_0(H_0, \omega) V_1 a_1(H_1, \omega) \cdots V_n a_n(H_n, \omega)\psi \, d\nu(\omega),$$

for $V_1, \dots, V_n \in \mathrm{op}$, $\psi \in \mathcal{H}^\infty$, converges as a Bochner integral in $\mathcal{H}^s$ for every $s \in \mathbb{R}$. This defines a well-defined map

$$T_\phi^{H_0,\dots,H_n} : \mathrm{op} \times \cdots \times \mathrm{op} \to \mathrm{op}\,.$$

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Suppose that we have $a_j(H_j, \omega) \in \mathrm{op}^0(\Theta)$ and

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If $\Theta = 1$, we get the usual MOI of [Peller, 2006]. If $\Theta = \sqrt{1 + \Delta}$ on $L^2(\mathbb{R}^d)$, then $\mathcal{H}^s = W_2^s(\mathbb{R}^d)$. For a spectral triple, take $\Theta = \sqrt{1 + D^2}$.

The main application:

An elliptic operator of order $r \in \mathbb{R}$ is an operator $H \in \mathrm{op}^r(\Theta)$ for which there exists a parametrix $P \in \mathrm{op}^{-r}(\Theta)$ such that

$$HP = 1_{\mathcal{H}^\infty} + R_1;$$
$$PH = 1_{\mathcal{H}^\infty} + R_2,$$

where $R_1, R_2 \in \mathrm{op}^{-\infty}$.
We call $f \in C^\infty(\mathbb{R})$ (or $\mathbb{R}_+$) of order $\beta \in \mathbb{R}$ if

$$(\sqrt{1+x^2})^{k-\beta+\epsilon} f^{(k)}(x)$$

is bounded for all $k \in \mathbb{N}$.
If f is of order β, if $H_0, \dots, H_n$ are symmetric and elliptic of order $h > 0$, and if $V_j \in \mathrm{op}^{r_j}$, then we obtain

$$T_{f^{[n]}}^{H_0,\dots,H_n}(V_1, \dots, V_n) \in \mathrm{op}^{(\beta-n)h+\sum r_j}.$$

E.g., if $f(x) = e^{-x}$, the MOI is a smoothing operator...

Some notes on elliptic operators:

Proposition

Let $H \in \mathrm{op}^r$ be an elliptic operator. If $x \in \mathcal{H}^{-\infty}$ is such that $Hx \in \mathcal{H}^s$ for an $s \in \mathbb{R}$, then $x \in \mathcal{H}^{s+r}$.

Proposition

Let $H \in \mathrm{op}^r$, $r \geqslant 0$, be an elliptic and symmetric operator. Then H is self-adjoint with domain $\mathcal{H}^r$.

Theorem

Let $H \in \mathrm{op}^r$, $r > 0$, be elliptic and symmetric, and let E denote its spectral measure. If $f \in L^\beta_\infty(E), \beta \in \mathbb{R}$, then $f(H) \in \mathrm{op}^{\beta r}$ with

$$\|f(H)\|_{\mathcal{H}^{s+\beta r} \to \mathcal{H}^s} \leqslant C_{s,H} \|f\|_{L^\beta_\infty(E)}.$$

Theorem (Expansion of MOIs)

Let f be of order β, H symmetric and elliptic of order $h > 0$, and $V_i \in \mathrm{op}^{r_i}$. If $\delta_H^n(V_i) \in \mathrm{op}^{n(h-\epsilon)+r_i}$ for all $n \in \mathbb{N}$, then we have

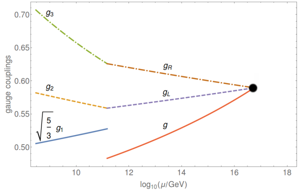
$$T_{f^{[n]}}^{H,\ldots,H}(V_1,\ldots,V_n) \sim \sum_{m=0}^{\infty} \sum_{m_1+\ldots+m_n=m} \frac{C_{m_1,\ldots,m_n}}{(n+m)!} \delta_H^{m_1}(V_1)\cdots\delta_H^{m_n}(V_n) f^{(n+m)}(H).$$

Two ways to expand the spectral action $\mathrm{Tr}(f(\frac{D+V}{\Lambda}))$:

1. Expand $\mathrm{Tr}(f(\frac{D+V}{\Lambda}))$ in Λ

$$\mathrm{Tr}(f(\frac{D_{M\times F}+V}{\Lambda})) = c_0\Lambda^4 \mathrm{vol}(M) + c_1\Lambda^2 \int R\sqrt{g}dx + c_2 \int \mathrm{tr} F_{\mu\nu}F^{\mu\nu} - c_3 \int |\phi|^2 + c_4 \int |\phi|^4 + \cdots$$

Spectral triple→Physical effective action, RG flow → measurable data. But: noncommutativity is ignored in intermediate step.



Renormalization Group flow

cf. [van Suijlekom, Chamseddine, Connes, JHEP]

2. Expand $\mathrm{Tr}(f(\frac{D+V}{\Lambda}))$ in $\Lambda^{-1}V$

   Taylor: $\mathrm{Tr}(f(\frac{D+V}{\Lambda})) = \sum_{n=0}^{\infty} \frac{\Lambda^{-n}}{n!} \mathrm{Tr}(T_{f^{[n]}}^{D/\Lambda,\ldots,D/\Lambda}(V,\ldots,V))$

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Combining this with $[f(\Theta), V] = T_{f^{[1]}}^{\Theta}([\Theta, V])$, one obtains formulas from [Connes-Moscovici,1995] like

$$[\Theta^\alpha, V] \sim \sum_{k=1}^{\infty} \binom{\alpha}{k} \delta_\Theta^k(V)\Theta^{\alpha-k}$$

and

$$[\log(\Theta), V] \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \delta_\Theta^k(V)\Theta^{-k}.$$

Besides old results, we also obtain a new one: an open question posed by Iochum a few years ago.

Corollary

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular s-summable spectral triple, $s > 0$. Let $V \in \mathcal{B}$ be self-adjoint and bounded, where $\mathcal{B}$ is the algebra generated by $\mathcal{A}$ and D. If

$$\mathrm{Tr}(Qe^{-tD^2})$$

admits an asymptotic expansion as $t \to 0$ for each $Q \in \mathcal{B}$, then

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Thanks!

Appendix

Proposition

Let $H \in \mathsf{op}^r(\Theta)$ be such that $[\Theta, H] \in \mathsf{op}^r$. If the extension

$$H : \mathcal{H}^{s_0+r} \to \mathcal{H}^{s_0}$$

has a bounded inverse

$$H^{-1} : \mathcal{H}^{s_0} \to \mathcal{H}^{s_0+r}$$

for one particular $s_0 \in \mathbb{R}$, then $H^{-1}\big|_{\mathcal{H}^\infty} \in \mathsf{op}^{-r}$. We have $HH^{-1}|_{\mathcal{H}^\infty} = H^{-1}H|_{\mathcal{H}^\infty} = 1_{\mathcal{H}^\infty}$. In particular, if $H \in \mathsf{op}^r$ and $[\Theta, H] \in \mathsf{op}^r$ with $r \geqslant 0$, then we have as (unbounded) operators

$$\sigma(H : \mathcal{H}^{s_0+r} \subseteq \mathcal{H}^{s_0} \to \mathcal{H}^{s_0}) = \sigma(H : \mathcal{H}^{s+r} \subseteq \mathcal{H}^{s} \to \mathcal{H}^{s})$$

for all $s \in \mathbb{R}$.

Lemma

Let $H \in \mathsf{op}^0$ be such that $[\Theta, H] \in \mathsf{op}^0$ and $\overline{H}^{0,0} : \mathcal{H} \to \mathcal{H}$ is self-adjoint. Then for all $s \in \mathbb{R}$, there is a constant $C_s > 0$ such that

$$\|(z-H)^{-1}\|_{\mathcal{H}^s \to \mathcal{H}^s} \leqslant C_s \frac{1}{|\Im(z)|} \left(\frac{\sqrt{1+|z|^2}}{|\Im(z)|} \right)^{2^{|s|}-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$