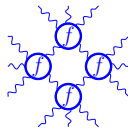
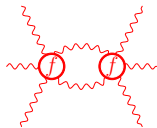
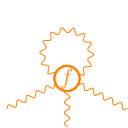


One-loop renormalizability in the spectral action using cyclic cocycles

Teun van Nuland

UNSW Sydney



Based on joint work with Walter van Suijlekom:

Cyclic cocycles in the spectral action (2022) JNCG

One-loop corrections of the spectral action (2022) JHEP

Cyclic cocycles and one-loop corrections in the spectral action (2023) Proc. of Symp. in Pure Math.

Part 1:

Cyclic expansion of the spectral action

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a $*$ -algebra \mathcal{A} and a self-adjoint operator D , both acting in the same Hilbert space \mathcal{H} , such that $(D - i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

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Other examples: Riemannian spin manifolds, Moyal plane

Over a manifold M , gauge fields are Lie-algebra-valued one-forms:

$$A = \sum_k a_k db_k \in \Omega_{dR}^1(M; \mathfrak{g})$$

for functions $a_k, b_k \in C^\infty(M; \mathfrak{g})$. Over a noncommutative space, gauge fields are self-adjoint elements of

$$\Omega_D^1(\mathcal{A}) := \left\{ \sum_k a_k [D, b_k] : a_k, b_k \in \mathcal{A} \right\} \subseteq \mathcal{B}(\mathcal{H}).$$

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a test function $f : \mathbb{R} \rightarrow \mathbb{R}$ the **spectral action** is given by

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Standard Model of Elementary Particles

Three generations of matter				Interactions (force carriers)	
Quarks		Leptons		Bosons	
I	II	I	II	I	II
u up	c charm	t top	e electron	g gluon	W W boson
d down	s strange	b bottom	μ muon	g gluon	Z Z boson
τ tau	ν _τ tau neutrino	ν _μ muon neutrino	ν _e electron neutrino	g gluon	W W boson
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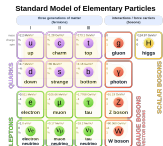
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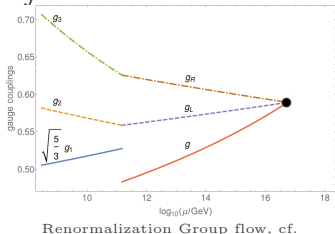
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→ Physical effective actions
 RG flow → physical parameters. But: non-rigorous and non-spectral



Taylor expansion of the spectral action:

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More generally:

$$\frac{1}{n!} \frac{d^n}{dt^n} \text{Tr}(f(D + t\Phi))|_{t=0} = \frac{1}{n} \sum_{i_1, \dots, i_n} f'[\lambda_{i_1}, \dots, \lambda_{i_n}] \Phi_{i_1, i_2} \cdots \Phi_{i_{n-1}, i_n} \Phi_{i_n, i_1}$$

where $f'[\lambda, \mu] = \frac{f'(\lambda) - f'(\mu)}{\lambda - \mu}$, $f'[\lambda, \mu, \nu] = \frac{f'[\lambda, \mu] - f'[\lambda, \nu]}{\mu - \nu}$, etc. are the divided differences of f' , $\{\psi_1, \psi_2, \dots\}$ is an eigenbasis of D with eigenvalues $\{\lambda_1, \lambda_2, \dots\}$, and $\Phi_{ij} = \langle \psi_i | \Phi \psi_j \rangle$.

Let $\{\psi_1, \psi_2, \dots\}$ be an eigenbasis of D with eigenvalues $\{\lambda_1, \lambda_2, \dots\}$.
 Write $\Phi_{ij} = \langle \psi_i | \Phi \psi_j \rangle$. Define

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 \text{Diagram: a central orange circle with four wavy lines extending outwards labeled } \Phi_1, \Phi_2, \Phi_3, \Phi_4 \text{ and a dashed line to } \Phi_n. & := \langle \Phi_1, \dots, \Phi_n \rangle \\
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 \end{aligned}$$

We summarize:

$$\begin{aligned}
 S[\Phi] - S[0] &= \sum_{n=0}^{\infty} \frac{1}{n} \langle \Phi, \dots, \Phi \rangle \\
 &= \text{Diagram: a central orange circle with one wavy line labeled } \Phi. + \frac{1}{2} \text{Diagram: a central blue circle with two wavy lines labeled } \Phi. + \frac{1}{3} \text{Diagram: a central red circle with three wavy lines labeled } \Phi. + \dots \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i_1, \dots, i_n} f'[\lambda_{i_1}, \dots, \lambda_{i_n}] \Phi_{i_1, i_2} \cdots \Phi_{i_{n-1}, i_n} \Phi_{i_n, i_1}
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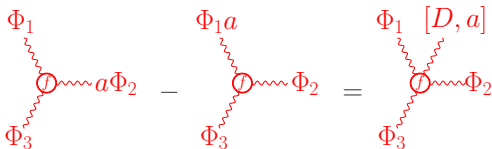
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From the two essential properties

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follows a beautiful relation between the brackets $\langle \cdot, \dots, \cdot \rangle$ and Connes' cyclic cohomology.

Denote by

$$\Omega_{\text{uni}}^n(\mathcal{A}) = \{a_0 \delta a_1 \cdots \delta a_n : a_i \in \mathcal{A}\}$$

the **universal n -forms** over \mathcal{A} . I.e., $\Omega_{\text{uni}}^0(\mathcal{A}) = \mathcal{A}$, and $\delta : \Omega_{\text{uni}}^n(\mathcal{A}) \rightarrow \Omega_{\text{uni}}^{n+1}(\mathcal{A})$ satisfies $\delta^2 = 0$ and the Leibniz rule:

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We have a representation $\pi : \Omega_{\text{uni}}^1(\mathcal{A}) \rightarrow \Omega_D^1(\mathcal{A})$ given by

$$\pi(a\delta b) := a[D, b].$$

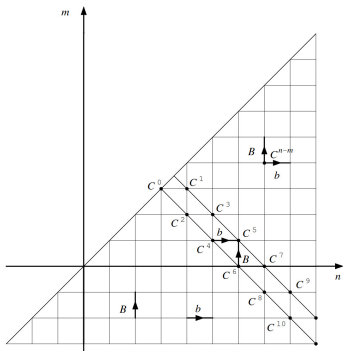


FIGURE 1. The (b, B) bicomplex

cf. [Connes, Academic Press '94 (III.1.γ)]

Cyclic cohomology extends Hochschild cohomology. It uses a bicomplex.

Even (resp. odd) cyclic cocycles are sequences $(\varphi_2, \varphi_4, \dots)$ (resp. $(\varphi_1, \varphi_3, \dots)$) of linear maps $\int_{\varphi_n} : \Omega_{\text{uni}}^n(\mathcal{A}) \rightarrow \mathbb{C}$.

We define

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n$$

$$:= \langle a_0 [D, a_1], [D, a_2], \dots, [D, a_n] \rangle .$$

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and, using $c_k := \frac{(-1)^{k-1} (k-1)!}{(2k-1)!}$, we define

$$\int_{\psi_{2k-1}} \omega := c_k \left(\int_{\phi_{2k-1}} \omega - \frac{1}{2} \int_{\phi_{2k}} \delta \omega \right).$$

Using the two essential properties of $\langle \cdot, \dots, \cdot \rangle$:

1. $\langle \Phi_1, \dots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \dots, \Phi_{n-1} \rangle$.
2. $\langle \Phi_1, a\Phi_2, \dots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \dots, \Phi_n a \rangle = \langle \Phi_1, [D, a], \Phi_2, \dots, \Phi_n \rangle$,

it follows that (ϕ_2, ϕ_4, \dots) and (ψ_1, ψ_3, \dots) form cyclic cocycles.

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Theorem (vN–van Suijlekom)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an s -summable spectral triple, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be nice enough. For all $\Phi = \pi_D(A) \in \Omega_D^1(\mathcal{A})_{\text{s.a.}}$:

$$\text{Tr}(f(D + \Phi) - f(D)) = \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}} \text{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F(A)^k \right),$$

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Here, the field strength or **curvature** $F(A) \in \Omega^2(\mathcal{A})$ of $A \in \Omega^1(\mathcal{A})$ is given by

$$F(A) := \delta A + A^2.$$

Example: If $A = a\delta b$, then $F(A) = \delta a\delta b + a\delta b a\delta b$ for $a, b \in \mathcal{A}$.

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Another important universal form is the **Chern–Simons** form

$$\mathrm{cs}_{2k+1}(A) := \int_0^1 A(t\delta A + t^2 A^2)^k dt \in \Omega^{2k+1}(\mathcal{A}).$$

Examples: $\mathrm{cs}_1(A) = A$, $\mathrm{cs}_3(A) = \frac{1}{2}A\delta A + \frac{1}{3}A^3$, etc.

A few words on the proof. Recall that

$$\langle \Phi_1, a\Phi_2, \dots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \dots, \Phi_n a \rangle = \langle \Phi_1, [D, a], \Phi_2, \dots, \Phi_n \rangle.$$

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We obtain, when $\Phi = a[D, b]$ and $A = a\delta b$,

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We obtain, when $\Phi = a[D, b]$ and $A = a\delta b$,

$$\begin{aligned}\langle a[D, b] \rangle &= \int_{\phi_1} A \\ \langle a[D, b], a[D, b] \rangle &= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A\end{aligned}$$

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We obtain, when $\Phi = a[D, b]$ and $A = a\delta b$,

$$\langle a[D, b] \rangle = \int_{\phi_1} A$$

$$\langle a[D, b], a[D, b] \rangle = \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A$$

$$\langle a[D, b], a[D, b], a[D, b] \rangle = \int_{\phi_3} A^3 + \int_{\phi_4} A\delta AA + \int_{\phi_5} A\delta A\delta A$$

etcetera.

Part 2:

One-Loop corrections to the Spectral Action

We recall

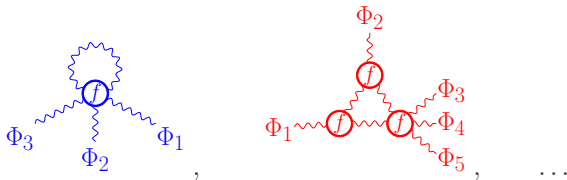
A central orange circle labeled f is connected to n external legs. The legs are wavy lines extending outwards, labeled $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \dots, \Phi_n$. The legs $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ are shown as solid wavy lines, while the leg Φ_n is shown as a dotted wavy line. To the right of the diagram is the definition $:= \langle \Phi_1, \dots, \Phi_n \rangle$.

Moreover:

$$\mathrm{Tr}(f(D + \Phi) - f(D)) = \sum_{n=0}^{\infty} \frac{1}{n} \langle \Phi, \dots, \Phi \rangle$$

The expansion of the trace formula is shown as a sum of Feynman diagrams. The first term is Φ connected to a circle labeled f by a wavy line. The second term is $\frac{1}{2} \Phi$ connected to a circle labeled f by two wavy lines. The third term is $\frac{1}{3} \Phi$ connected to a circle labeled f by three wavy lines. The diagrams are colored: orange for the first term, blue for the second, and red for the third. The expansion ends with $+ \dots$.

We wish to incorporate more Feynman diagrams, like



according to

$$\int_{H_N} e^{-\text{Tr}(f(D+\Phi))} d[\Phi].$$

By employing random matrix theory, we can construct Feynman diagrams by e.g.,

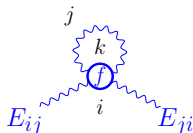
$$\begin{array}{c} V_n \\ \vdots \\ \text{---} G_1 \text{---} \\ \vdots \\ V_1 \end{array} \text{---} \begin{array}{c} V_{n+1} \\ \vdots \\ \text{---} G_2 \text{---} \\ \vdots \\ V_m \end{array} = \frac{-1}{Z[0]} \int_{H_N} \left(\begin{array}{c} V_n \\ \vdots \\ \text{---} G_1 \text{---} \\ \vdots \\ V_1 \end{array} \text{---} Q \text{---} \begin{array}{c} V_{n+1} \\ \vdots \\ \text{---} G_2 \text{---} \\ \vdots \\ V_m \end{array} \right) e^{-\frac{1}{2} \langle Q, Q \rangle} dQ.$$

The Feynman rules are derived: an edge bordering i and j adds a factor $\frac{1}{f'[\lambda_i, \lambda_j]} = \frac{\lambda_i - \lambda_j}{f'(\lambda_i) - f'(\lambda_j)}$. Same Feynman rules as in

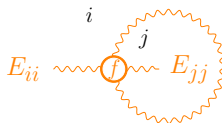
[\[Belliard–Charbonnier–Eynard–Garcia-Failde, '21\]!](#)

As such we can define all diagrams with noncommutative vertices of arbitrary valence, their non-locality modulated by f .

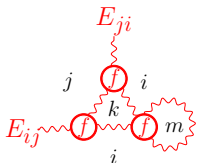
Examples:



$$= \sum_{k=1}^N \frac{f'[i, j, k, j]}{f'[j, k]}$$



$$= \frac{f'[i, j, j, i]}{f'[i, j]}$$



$$= \sum_{k,m=1}^N \frac{f'[j, i, k] f'[k, i, m, i] f'[j, k, i]}{f'[j, k] f'[i, k]^2}$$

We let

$$\langle\langle \Phi_1, \dots, \Phi_n \rangle\rangle_N^{1L}$$

be the sum of all relevant one-loop one-particle-irreducible n -point functions, whose external edges can naturally be labeled cyclically. The **one-loop quantum effective spectral action** is defined to be the formal series

$$\sum_{n=1}^{\infty} \frac{1}{n} \langle\langle \Phi, \dots, \Phi \rangle\rangle_N^{1L}.$$

To note a few terms,

$$\langle\langle \Phi_1, \dots, \Phi_n \rangle\rangle_N^{1L} =$$

The equation shows the first two terms of a series expansion. The first term is a blue diagram with a central vertex labeled 'f' inside a circle. It has three external wavy lines: one to the left labeled Φ_n , one to the right labeled Φ_1 , and one to the top labeled Φ_2 . The top line is a solid wavy line, while the left and right lines are dashed. The second term is a red diagram with two vertices labeled 'f' inside circles. It has four external wavy lines: one to the left labeled Φ_1 , one to the top labeled Φ_2 , one to the right labeled Φ_3 , and one to the bottom labeled Φ_n . The top and right lines are solid, while the left and bottom lines are dashed. The two vertices are connected by a wavy line. The diagram is followed by a plus sign and an ellipsis.

By definition, $\langle\langle \Phi_1, \dots, \Phi_n \rangle\rangle_N^{1L} = \langle\langle \Phi_2, \dots, \Phi_n, \Phi_1 \rangle\rangle_N^{1L}$.

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$$\langle\langle \Phi_1, \dots, \Phi_n \rangle\rangle_N^{1L} = \text{diagram 1} + \text{diagram 2} + \dots$$

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Question: does

$$\langle\langle \Phi_1, a\Phi_2, \dots, \Phi_n \rangle\rangle_N^{1L} - \langle\langle \Phi_1 a, \Phi_2, \dots, \Phi_n a \rangle\rangle_N^{1L} = \langle\langle \Phi_1, [D, a], \Phi_2, \dots, \Phi_n \rangle\rangle_N^{1L}$$

hold as well?

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Answer: yes.

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hold as well?

Answer: yes. cf. [vN–van Suijlekom, PoSiPM '23]

For example, the contributions to $\langle\langle a\Phi_1, \Phi_2 \rangle\rangle_N^{1L} - \langle\langle \Phi_1, \Phi_2 a \rangle\rangle_N^{1L}$ are

$$a\Phi_1 \text{---} \text{---} \text{---} \Phi_2 - \Phi_1 \text{---} \text{---} \text{---} \Phi_2 a = \Phi_1 \text{---} \text{---} \text{---} \Phi_2 + \Phi_1 \text{---} \text{---} \text{---} \Phi_2 + \Phi_1 \text{---} \text{---} \text{---} \Phi_2$$

$[D, a]$
 $[D, a]$
 $[D, a]$

and

$$a\Phi_1 \text{---} \text{---} \Phi_2 - \Phi_1 \text{---} \text{---} \Phi_2 a = \Phi_1 \text{---} \text{---} \Phi_2$$

$[D, a]$

and

$$\Phi_2 \text{---} \text{---} a\Phi_1 - \Phi_2 a \text{---} \text{---} \Phi_1 = \Phi_1 \text{---} \text{---} \Phi_2 + \Phi_1 \text{---} \text{---} \Phi_2 + \Phi_1 \text{---} \text{---} \Phi_2$$

$[D, a]$
 $[D, a]$
 $[D, a]$

We derive

$$\langle\langle a\Phi_1, \Phi_2 \rangle\rangle_N^{1L} - \langle\langle \Phi_1, \Phi_2 a \rangle\rangle_N^{1L} = D, \mathcal{D} \langle a \langle \Phi_1, \Phi_2 \rangle \rangle_N^{1L}.$$

As the non-analytic part of our earlier theorem only depended on the cyclicity and commutation property of $\langle \cdots \rangle$, we conclude that the **one-loop quantum effective spectral action** takes the exact same form as the spectral action.

Theorem (vN–van Suijlekom)

There exist cyclic cocycles $(\psi_1^N, \psi_3^N, \dots)$ and $(\phi_2^N, \phi_4^N, \dots)$ such that for all ‘finite-dimensional’ $\Phi = \pi_D(A) \in \Omega_D^1(\mathcal{A})_{\text{s.a.}}$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \langle\langle \Phi, \dots, \Phi \rangle\rangle_N^{1L} \sim \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}^N} \text{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}^N} F(A)^k \right).$$

We can therefore absorb all one-loop divergences into the cyclic cocycles!

Open questions:

1. Can we do the same for higher loop?
2. Can we describe the renormalisation group flow of these cyclic cocycles?
3. Can we replace H_N by a subspace $\Omega_D^1(\mathcal{A})_{sa}$ (modulo gauge transformations)?
4. Can we treat the non-compact case (using multiple operator integration?)?
5. How to understand the difference between f Schwartz and f polynomial? Is there still a relation to TR in the more general case?

Help welcome!

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Thanks for attention :)