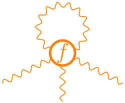
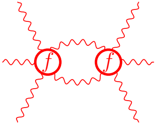
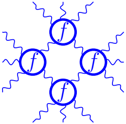
One-loop renormalizability in the spectral action using cyclic cocycles

Teun van Nuland

UNSW Sydney







Based on joint work with Walter van Suijlekom:

Cyclic cocycles in the spectral action (2022) JNCG

One-loop corrections of the spectral action (2022) JHEP

Cyclic cocycles and one-loop corrections in the spectral action (2023) Proc. of Symp. in Pure Math.

Part 1:

Cyclic expansion of the spectral action

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a *-algebra $\mathcal{A}$ and a self-adjoint operator D, both acting in the same Hilbert space $\mathcal{H}$, such that $(D - i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

Example: spectral triple associated to $M = S^1$

Algebra: $\mathcal{A} := C^\infty(S^1)$

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a *-algebra $\mathcal{A}$ and a self-adjoint operator D, both acting in the same Hilbert space $\mathcal{H}$, such that $(D - i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

`Example:` spectral triple associated to $M = S^1$
Algebra: $\mathcal{A} := C^\infty(S^1)$
Hilbert space: $\mathcal{H} := L^2(S^1)$ with basis $\{\psi_k\}_{k \in \mathbb{Z}}$, $\psi_k(\theta) = e^{ik\theta}$.

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a *-algebra $\mathcal{A}$ and a self-adjoint operator D, both acting in the same Hilbert space $\mathcal{H}$, such that $(D-i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

Example: spectral triple associated to $M = S^1$
Algebra: $\mathcal{A} := C^\infty(S^1)$
Hilbert space: $\mathcal{H} := L^2(S^1)$ with basis $\{\psi_k\}_{k\in\mathbb{Z}}$, $\psi_k(\theta) = e^{ik\theta}$.
$\mathcal{A} \times \mathcal{H} \to \mathcal{H}$, $(g \cdot \psi)(\theta) := g(\theta)\psi(\theta)$.

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a *-algebra $\mathcal{A}$ and a self-adjoint operator D, both acting in the same Hilbert space $\mathcal{H}$, such that $(D-i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

Example: spectral triple associated to $M = S^1$
Algebra: $\mathcal{A} := C^\infty(S^1)$
Hilbert space: $\mathcal{H} := L^2(S^1)$ with basis $\{\psi_k\}_{k\in\mathbb{Z}}$, $\psi_k(\theta) = e^{ik\theta}$.
$\mathcal{A} \times \mathcal{H} \to \mathcal{H}$, $(g \cdot \psi)(\theta) := g(\theta)\psi(\theta)$.
Dirac operator: $D := -i\frac{d}{d\theta}$,

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a *-algebra $\mathcal{A}$ and a self-adjoint operator D, both acting in the same Hilbert space $\mathcal{H}$, such that $(D - i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

Example: spectral triple associated to $M = S^1$
Algebra: $\mathcal{A} := C^\infty(S^1)$
Hilbert space: $\mathcal{H} := L^2(S^1)$ with basis $\{\psi_k\}_{k\in\mathbb{Z}}$, $\psi_k(\theta) = e^{ik\theta}$.
$\mathcal{A} \times \mathcal{H} \to \mathcal{H}$, $(g \cdot \psi)(\theta) := g(\theta)\psi(\theta)$.
Dirac operator: $D := -i\frac{d}{d\theta}$, $D\psi_k = k\psi_k$

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a $*$-algebra $\mathcal{A}$ and a self-adjoint operator D, both acting in the same Hilbert space $\mathcal{H}$, such that $(D-i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

Example: spectral triple associated to $M = S^1$

Algebra: $\mathcal{A} := C^\infty(S^1)$

Hilbert space: $\mathcal{H} := L^2(S^1)$ with basis $\{\psi_k\}_{k\in\mathbb{Z}}$, $\psi_k(\theta) = e^{ik\theta}$.

$\mathcal{A} \times \mathcal{H} \to \mathcal{H}$, $(g \cdot \psi)(\theta) := g(\theta)\psi(\theta)$.

Dirac operator: $D := -i\frac{d}{d\theta}$, $D\psi_k = k\psi_k$

$[D, a]\psi = D(a \cdot \psi) - a \cdot D(\psi) = (-i)\frac{d}{d\theta}(a \cdot \psi) - (-i)a\frac{d}{d\theta}\psi = (-i)\frac{da}{d\theta}\psi$

$$\boxed{[D, a] = -i\frac{da}{d\theta}}$$

For more general commutative spectral triples, D encodes the metric structure of M, and $[D, a]$ is related to the first derivatives of a.

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a *-algebra $\mathcal{A}$ and a self-adjoint operator D, both acting in the same Hilbert space $\mathcal{H}$, such that $(D-i)^{-1}$ is compact and such that $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$.

Example: spectral triple associated to $M = S^1$

Algebra: $\mathcal{A} := C^\infty(S^1)$

Hilbert space: $\mathcal{H} := L^2(S^1)$ with basis $\{\psi_k\}_{k\in\mathbb{Z}}$, $\psi_k(\theta) = e^{ik\theta}$.

$\mathcal{A} \times \mathcal{H} \to \mathcal{H}$, $(g \cdot \psi)(\theta) := g(\theta)\psi(\theta)$.

Dirac operator: $D := -i\frac{d}{d\theta}$, $D\psi_k = k\psi_k$

$[D, a]\psi = D(a\cdot\psi) - a\cdot D(\psi) = (-i)\frac{d}{d\theta}(a\cdot\psi) - (-i)a\frac{d}{d\theta}\psi = (-i)\frac{da}{d\theta}\psi$

$$\boxed{[D, a] = -i\frac{da}{d\theta}}$$

For more general commutative spectral triples, D encodes the metric structure of M, and $[D, a]$ is related to the first derivatives of a.

Other examples: Riemannian spin manifolds, Moyal plane

Over a manifold M, gauge fields are Lie-algebra-valued one-forms:

$$A = \sum_k a_k db_k \in \Omega^1_{dR}(M; \mathfrak{g})$$

for functions $a_k, b_k \in C^\infty(M; \mathfrak{g})$. Over a noncommutative space, gauge fields are self-adjoint elements of

$$\Omega^1_D(\mathcal{A}) := \left\{ \sum_k a_k [D, b_k] \,:\, a_k, b_k \in \mathcal{A} \right\} \subseteq \mathcal{B}(\mathcal{H}) \,.$$

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a test function $f : \mathbb{R} \to \mathbb{R}$ the spectral action is given by

$$S[\Phi] = \mathrm{Tr}(f(D + \Phi)).$$

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a test function $f : \mathbb{R} \to \mathbb{R}$ the spectral action is given by

$$S[\Phi] = \mathrm{Tr}(f(D + \Phi)).$$

If $\Phi = \sum a_j [D, b_j] \in \Omega^1_D(\mathcal{A})_{\mathrm{s.a.}}$, then $(\mathcal{A}, \mathcal{H}, D + \Phi)$ is again a spectral triple.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a test function $f : \mathbb{R} \to \mathbb{R}$ the spectral action is given by

$$S[\Phi] = \mathrm{Tr}(f(D + \Phi)).$$

If $\Phi = \sum a_j[D, b_j] \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$, then $(\mathcal{A}, \mathcal{H}, D + \Phi)$ is again a spectral triple.
For the right choice of $(\mathcal{A}, \mathcal{H}, D) = (C^\infty(M) \otimes F, \mathcal{H}, D_{M\times F})$,
$\mathrm{Tr}(f(D + \Phi))$ produces the full (bosonic part of the) standard model

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a test function $f : \mathbb{R} \to \mathbb{R}$ the spectral action is given by

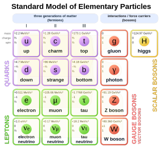
$$S[\Phi] = \mathrm{Tr}(f(D + \Phi)).$$

If $\Phi = \sum a_j [D, b_j] \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$, then $(\mathcal{A}, \mathcal{H}, D + \Phi)$ is again a spectral triple.

For the right choice of $(\mathcal{A}, \mathcal{H}, D) = (C^\infty(M) \otimes F, \mathcal{H}, D_{M\times F})$, $\mathrm{Tr}(f(D + \Phi))$ produces the full (bosonic part of the) standard model

$$\mathrm{Tr}\left(f\left(\frac{D_{M\times F} + \Phi}{\Lambda}\right)\right) \sim c_0 \Lambda^4 \mathrm{vol}(M) + c_1 \Lambda^2 \int R\sqrt{g}dx + c_2 \int \mathrm{tr} F_{\mu\nu} F^{\mu\nu}$$

$$- c_3 \int |\phi|^2 + c_4 \int |\phi|^4 + \cdots$$



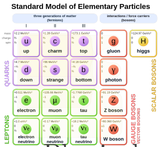
Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a test function $f : \mathbb{R} \to \mathbb{R}$ the spectral action is given by

$$S[\Phi] = \mathrm{Tr}(f(D + \Phi)).$$

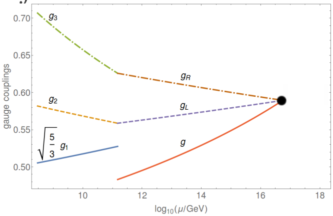
If $\Phi = \sum a_j [D, b_j] \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$, then $(\mathcal{A}, \mathcal{H}, D + \Phi)$ is again a spectral triple.

For the right choice of $(\mathcal{A}, \mathcal{H}, D) = (C^\infty(M) \otimes F, \mathcal{H}, D_{M\times F})$, $\mathrm{Tr}(f(D + \Phi))$ produces the full (bosonic part of the) standard model

$$\mathrm{Tr}\left(f\left(\frac{D_{M\times F} + \Phi}{\Lambda}\right)\right) \sim c_0 \Lambda^4 \mathrm{vol}(M) + c_1 \Lambda^2 \int R\sqrt{g}dx + c_2 \int \mathrm{tr} F_{\mu\nu} F^{\mu\nu} - c_3 \int |\phi|^2 + c_4 \int |\phi|^4 + \cdots$$



→Physical effective actions
RG flow → physical parameters. But: non-rigorous and non-spectral



Renormalization Group flow, cf.

[van Suijlekom-Chamseddine-Connes, JHEP]

Taylor expansion of the spectral action:

$$S[\Phi] - S[0] = \sum_{n=1}^{\infty} \frac{d^n}{dt^n} \operatorname{Tr}(f(D + t\Phi))\Big|_{t=0}$$

Taylor expansion of the spectral action:

$$S[\Phi] - S[0] = \sum_{n=1}^{\infty} \frac{d^n}{dt^n} \operatorname{Tr}(f(D + t\Phi))\Big|_{t=0}$$

For instance, if $f(x) = x^4$:

$$\frac{d}{dt} \operatorname{Tr}((D + t\Phi)^4)\Big|_{t=0} = \operatorname{Tr}(4D^3\Phi)$$

Taylor expansion of the spectral action:

$$S[\Phi] - S[0] = \sum_{n=1}^{\infty} \frac{d^n}{dt^n} \operatorname{Tr}(f(D + t\Phi))\Big|_{t=0}$$

For instance, if $f(x) = x^4$:

$$\frac{d}{dt} \operatorname{Tr}((D + t\Phi)^4)\Big|_{t=0} = \operatorname{Tr}(4D^3\Phi)$$

More generally:

$$\frac{d}{dt} \operatorname{Tr}(f(D + t\Phi))\Big|_{t=0} = \operatorname{Tr}(f'(D)\Phi)$$

Taylor expansion of the spectral action:

$$S[\Phi] - S[0] = \sum_{n=1}^{\infty} \frac{d^n}{dt^n} \operatorname{Tr}(f(D + t\Phi))\Big|_{t=0}$$

For instance, if $f(x) = x^4$:

$$\frac{d}{dt} \operatorname{Tr}((D + t\Phi)^4)\Big|_{t=0} = \operatorname{Tr}(4D^3\Phi)$$

More generally:

$$\frac{d}{dt} \operatorname{Tr}(f(D + t\Phi))\Big|_{t=0} = \operatorname{Tr}(f'(D)\Phi)$$

If $f(x) = x^4$:

$$\frac{d^2}{dt^2} \operatorname{Tr}((D + t\Phi)^4)\Big|_{t=0} = \operatorname{Tr}(16\Phi^2 D^2 + 8\Phi D \Phi D)$$

Taylor expansion of the spectral action:

$$S[\Phi] - S[0] = \sum_{n=1}^{\infty} \frac{d^n}{dt^n} \operatorname{Tr}(f(D + t\Phi))\Big|_{t=0}$$

For instance, if $f(x) = x^4$:

$$\frac{d}{dt} \operatorname{Tr}((D + t\Phi)^4)\Big|_{t=0} = \operatorname{Tr}(4D^3\Phi)$$

More generally:

$$\frac{d}{dt} \operatorname{Tr}(f(D + t\Phi))\Big|_{t=0} = \operatorname{Tr}(f'(D)\Phi)$$

If $f(x) = x^4$:

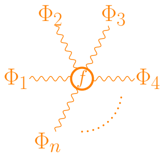
$$\frac{d^2}{dt^2} \operatorname{Tr}((D + t\Phi)^4)\Big|_{t=0} = \operatorname{Tr}(16\Phi^2 D^2 + 8\Phi D\Phi D)$$

More generally:

$$\frac{1}{n!} \frac{d^n}{dt^n} \operatorname{Tr}(f(D + t\Phi))|_{t=0} = \frac{1}{n} \sum_{i_1,\ldots,i_n} f'[\lambda_{i_1}, \ldots, \lambda_{i_n}]\Phi_{i_1,i_2} \cdots \Phi_{i_{n-1},i_n}\Phi_{i_n,i_1}$$

where $f'[\lambda, \mu] = \frac{f'(\lambda) - f'(\mu)}{\lambda - \mu}$, $f'[\lambda, \mu, \nu] = \frac{f'[\lambda,\mu] - f'[\lambda,\nu]}{\mu - \nu}$, etc. are the divided differences of f', $\{\psi_1, \psi_2, \ldots\}$ is an eigenbasis of D with eigenvalues $\{\lambda_1, \lambda_2, \ldots\}$, and $\Phi_{ij} = \langle \psi_i | \Phi \psi_j \rangle$.

Let $\{\psi_1, \psi_2, \dots\}$ be an eigenbasis of D with eigenvalues $\{\lambda_1, \lambda_2, \dots\}$. Write $\Phi_{ij} = \langle \psi_i | \Phi \psi_j \rangle$. Define



$$:= \quad \langle \Phi_1, \dots, \Phi_n \rangle$$

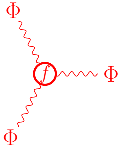
$$:= \sum_{i_1,\dots,i_n} f'[\lambda_{i_1}, \dots, \lambda_{i_n}] (\Phi_1)_{i_1,i_2} \cdots (\Phi_{n-1})_{i_{n-1},i_n} (\Phi_n)_{i_n,i_1}$$

We summarize:

$$S[\Phi] - S[0] = \sum_{n=0}^{\infty} \frac{1}{n} \langle \Phi, \dots, \Phi \rangle$$

$=$ 

$+ \frac{1}{2}$ 

$+ \frac{1}{3}$ 

$+ \dots$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i_1,\dots,i_n} f'[\lambda_{i_1}, \dots, \lambda_{i_n}] \Phi_{i_1,i_2} \cdots \Phi_{i_{n-1},i_n} \Phi_{i_n,i_1}$$

$$\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$$

$$\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$$

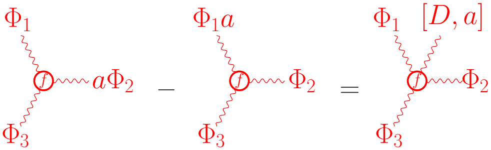
Moreover,

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$$

$$\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$$

Moreover,

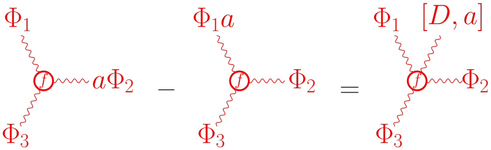
$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$$



$$\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$$

Moreover,

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$$

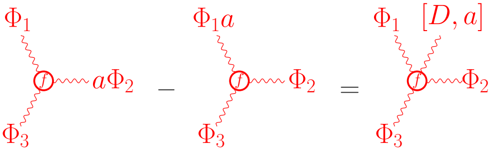


$$\langle \Phi, a\Phi' \rangle - \langle \Phi a, \Phi' \rangle = \sum_{i,j,k} f'[\lambda_i, \lambda_j](\Phi_{ij} a_{jk} \Phi'_{ki} - \Phi_{ik} a_{kj} \Phi'_{ji})$$

$$\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$$

Moreover,

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$$

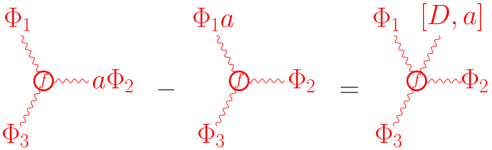


$$\begin{aligned} \langle \Phi, a\Phi' \rangle - \langle \Phi a, \Phi' \rangle &= \sum_{i,j,k} f'[\lambda_i, \lambda_j](\Phi_{ij} a_{jk} \Phi'_{ki} - \Phi_{ik} a_{kj} \Phi'_{ji}) \\ &= \sum_{i,j,k} (f'[\lambda_i, \lambda_j,] - f'[\lambda_i, \lambda_k]) \Phi_{ij} a_{jk} \Phi'_{ki} \end{aligned}$$

$$\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$$

Moreover,

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$$

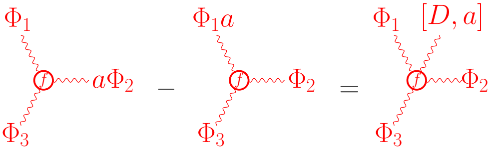


$$\begin{aligned}
\langle \Phi, a\Phi' \rangle - \langle \Phi a, \Phi' \rangle &= \sum_{i,j,k} f'[\lambda_i, \lambda_j](\Phi_{ij} a_{jk} \Phi'_{ki} - \Phi_{ik} a_{kj} \Phi'_{ji}) \\
&= \sum_{i,j,k} (f'[\lambda_i, \lambda_j,] - f'[\lambda_i, \lambda_k]) \Phi_{ij} a_{jk} \Phi'_{ki} \\
&= \sum_{i,j,k} f'[\lambda_i, \lambda_j, \lambda_k](\lambda_j - \lambda_k) \Phi_{ij} a_{jk} \Phi'_{ki}
\end{aligned}$$

$$\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$$

Moreover,

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$$

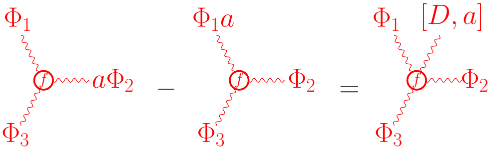


$$\begin{aligned}
\langle \Phi, a\Phi' \rangle - \langle \Phi a, \Phi' \rangle &= \sum_{i,j,k} f'[\lambda_i, \lambda_j](\Phi_{ij} a_{jk} \Phi'_{ki} - \Phi_{ik} a_{kj} \Phi'_{ji}) \\
&= \sum_{i,j,k} (f'[\lambda_i, \lambda_j,] - f'[\lambda_i, \lambda_k]) \Phi_{ij} a_{jk} \Phi'_{ki} \\
&= \sum_{i,j,k} f'[\lambda_i, \lambda_j, \lambda_k](\lambda_j - \lambda_k) \Phi_{ij} a_{jk} \Phi'_{ki} \\
&= \sum_{i,j,k} f'[\lambda_i, \lambda_j, \lambda_k] \Phi_{ij} [D, a]_{jk} \Phi'_{ki}
\end{aligned}$$

$$\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$$

Moreover,

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$$



$$\begin{aligned}
\langle \Phi, a\Phi' \rangle - \langle \Phi a, \Phi' \rangle &= \sum_{i,j,k} f'[\lambda_i, \lambda_j](\Phi_{ij} a_{jk} \Phi'_{ki} - \Phi_{ik} a_{kj} \Phi'_{ji}) \\
&= \sum_{i,j,k} (f'[\lambda_i, \lambda_j,] - f'[\lambda_i, \lambda_k]) \Phi_{ij} a_{jk} \Phi'_{ki} \\
&= \sum_{i,j,k} f'[\lambda_i, \lambda_j, \lambda_k](\lambda_j - \lambda_k) \Phi_{ij} a_{jk} \Phi'_{ki} \\
&= \sum_{i,j,k} f'[\lambda_i, \lambda_j, \lambda_k] \Phi_{ij} [D, a]_{jk} \Phi'_{ki} \\
&= \langle \Phi, [D, a], \Phi' \rangle
\end{aligned}$$

From the two essential properties

1. $\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$.
2. $\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$,

follows a beautiful relation between the brackets $\langle ., \ldots, . \rangle$ and Connes' cyclic cohomology.

Denote by

$$\Omega^n_{\mathrm{uni}}(\mathcal{A}) = \{a_0 \delta a_1 \cdots \delta a_n : \ a_i \in \mathcal{A}\}$$

the universal n-forms over $\mathcal{A}$. I.e., $\Omega^0_{\mathrm{uni}}(\mathcal{A}) = \mathcal{A}$, and $\delta : \Omega^n_{\mathrm{uni}}(\mathcal{A}) \to \Omega^{n+1}_{\mathrm{uni}}(\mathcal{A})$ satisfies $\delta^2 = 0$ and the Leibniz rule:

$$\delta(ab) = a(\delta b) + (\delta a)b.$$

From the two essential properties

1. $\langle \Phi_1, \dots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \dots, \Phi_{n-1} \rangle$.
2. $\langle \Phi_1, a\Phi_2, \dots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \dots, \Phi_n a \rangle = \langle \Phi_1, [D,a], \Phi_2, \dots, \Phi_n \rangle$,

follows a beautiful relation between the brackets $\langle ., \dots, . \rangle$ and Connes' cyclic cohomology.

Denote by

$$\Omega^n_{\mathrm{uni}}(\mathcal{A}) = \{a_0 \delta a_1 \cdots \delta a_n : \ a_i \in \mathcal{A}\}$$

the universal n-forms over $\mathcal{A}$. I.e., $\Omega^0_{\mathrm{uni}}(\mathcal{A}) = \mathcal{A}$, and $\delta : \Omega^n_{\mathrm{uni}}(\mathcal{A}) \to \Omega^{n+1}_{\mathrm{uni}}(\mathcal{A})$ satisfies $\delta^2 = 0$ and the Leibniz rule:

$$\delta(ab) = a(\delta b) + (\delta a)b.$$

We have a representation $\pi : \Omega^1_{\mathrm{uni}}(\mathcal{A}) \to \Omega^1_D(\mathcal{A})$ given by

$$\pi(a\delta b) := a[D,b].$$

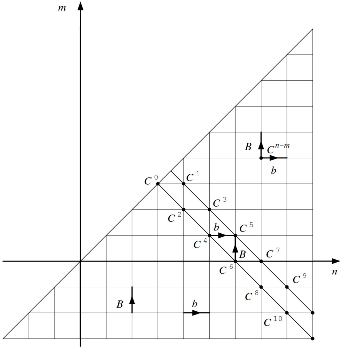


FIGURE 1. The (b, B) bicomplex

cf. [Connes, Academic Press '94 (III.1.γ)]

Cyclic cohomology extends Hochschild cohomology. It uses a bicomplex. Even (resp. odd) cyclic cocycles are sequences $(\varphi_2, \varphi_4 \ldots)$ (resp. $(\varphi_1, \varphi_3, \ldots)$) of linear maps $\int_{\varphi_n} : \Omega^n_{\text{uni}}(\mathcal{A}) \to \mathbb{C}$.

We define

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n$$

$$:= \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n] \rangle \,.$$

We define

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n := \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n] \rangle ,$$

and, using $c_k := \frac{(-1)^{k-1}(k-1)!}{(2k-1)!}$, we define

$$\int_{\psi_{2k-1}} \omega := c_k \left(\int_{\phi_{2k-1}} \omega - \tfrac{1}{2} \int_{\phi_{2k}} \delta\omega \right).$$

Using the two essential properties of $\langle ., \ldots, . \rangle$:

1. $\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$.
2. $\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$,

it follows that $(\phi_2, \phi_4, \ldots)$ and $(\psi_1, \psi_3, \ldots)$ form cyclic cocycles.

We define

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n := \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n] \rangle ,$$

and, using $c_k := \frac{(-1)^{k-1}(k-1)!}{(2k-1)!}$, we define

$$\int_{\psi_{2k-1}} \omega := c_k (\int_{\phi_{2k-1}} \omega - \tfrac{1}{2} \int_{\phi_{2k}} \delta\omega).$$

Using the two essential properties of $\langle ., \ldots, . \rangle$:

1. $\langle \Phi_1, \ldots, \Phi_n \rangle = \langle \Phi_n, \Phi_1, \ldots, \Phi_{n-1} \rangle$.
2. $\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle$,

it follows that $(\phi_2, \phi_4, \ldots)$ and $(\psi_1, \psi_3, \ldots)$ form cyclic cocycles.

Theorem (vN–van Suijlekom)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an s-summable spectral triple, and let $f : \mathbb{R} \to \mathbb{R}$ be nice enough. For all $\Phi = \pi_D(A) \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$:

$$\operatorname{Tr}(f(D+\Phi) - f(D)) = \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}} \operatorname{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F(A)^k \right),$$

where $c_k := \frac{(2k-1)!}{(-1)^{k-1}(k-1)!}$. This series converges absolutely.

Theorem (vN–van Suijlekom)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an s-summable spectral triple, and let $f : \mathbb{R} \to \mathbb{R}$ be nice enough. For all $\Phi = \pi_D(A) \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$:

$$\mathrm{Tr}(f(D+\Phi) - f(D)) = \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F(A)^k \right),$$

where $c_k := \frac{(2k-1)!}{(-1)^{k-1}(k-1)!}$. This series converges absolutely.

Here, the field strength or curvature $F(A) \in \Omega^2(\mathcal{A})$ of $A \in \Omega^1(\mathcal{A})$ is given by

$$F(A) := \delta A + A^2.$$

Example: If $A = a\delta b$, then $F(A) = \delta a \delta b + a\delta b\, a\delta b$ for $a, b \in \mathcal{A}$.

Theorem (vN–van Suijlekom)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an s-summable spectral triple, and let $f : \mathbb{R} \to \mathbb{R}$ be nice enough. For all $\Phi = \pi_D(A) \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$:

$$\operatorname{Tr}(f(D + \Phi) - f(D)) = \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \tfrac{1}{2k} \int_{\phi_{2k}} F(A)^k \right),$$

where $c_k := \frac{(2k-1)!}{(-1)^{k-1}(k-1)!}$. This series converges absolutely.

Here, the field strength or curvature $F(A) \in \Omega^2(\mathcal{A})$ of $A \in \Omega^1(\mathcal{A})$ is given by

$$F(A) := \delta A + A^2.$$

Example: If $A = a\delta b$, then $F(A) = \delta a \delta b + a\delta b\, a\delta b$ for $a, b \in \mathcal{A}$.

Another important universal form is the Chern–Simons form

$$\mathrm{cs}_{2k+1}(A) := \int_0^1 A(t\delta A + t^2 A^2)^k dt \in \Omega^{2k+1}(\mathcal{A})\,.$$

Examples: $\mathrm{cs}_1(A) = A$, $\mathrm{cs}_3(A) = \frac{1}{2} A\delta A + \frac{1}{3} A^3$, etc.

A few words on the proof. Recall that

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle .$$

A few words on the proof. Recall that

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle = \langle \Phi_1, [D,a], \Phi_2, \ldots, \Phi_n \rangle .$$

We obtain, when $\Phi = a[D,b]$ and $A = a\delta b$,

$$\langle a[D,b] \rangle = \int_{\phi_1} A$$

A few words on the proof. Recall that

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle = \langle \Phi_1, [D,a], \Phi_2, \ldots, \Phi_n \rangle \,.$$

We obtain, when $\Phi = a[D,b]$ and $A = a\delta b$,

$$\langle a[D,b] \rangle = \int_{\phi_1} A$$

$$\langle a[D,b], a[D,b] \rangle = \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A$$

A few words on the proof. Recall that

$$\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle - \langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle = \langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle .$$

We obtain, when $\Phi = a[D, b]$ and $A = a\delta b$,

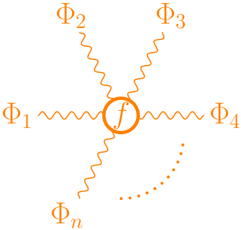
$$\begin{aligned} \langle a[D,b] \rangle &= \int_{\phi_1} A \\ \langle a[D,b], a[D,b] \rangle &= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A \\ \langle a[D,b], a[D,b], a[D,b] \rangle &= \int_{\phi_3} A^3 + \int_{\phi_4} A\delta A A + \int_{\phi_5} A\delta A\delta A \end{aligned}$$

etcetera.

Part 2:

One-Loop corrections to the Spectral Action

We recall



$$:= \quad \langle \Phi_1, \ldots, \Phi_n \rangle.$$

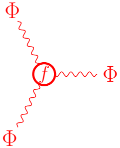
Moreover:

$$\mathrm{Tr}(f(D+\Phi) - f(D)) = \sum_{n=0}^{\infty} \frac{1}{n} \langle \Phi, \ldots, \Phi \rangle$$

$=$

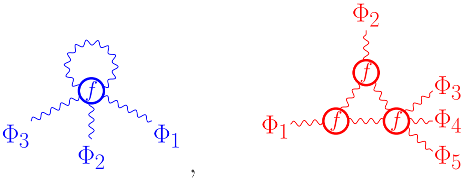
$+ \frac{1}{2}$

$+ \frac{1}{3}$



$+ \ldots.$

We wish to incorporate more Feynman diagrams, like

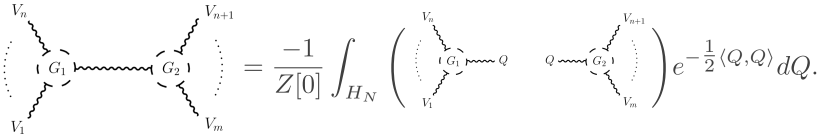


, , …

according to

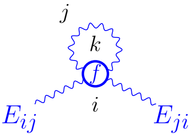
$$\int_{H_N} e^{-\operatorname{Tr}(f(D+\Phi))} d[\Phi].$$

By employing random matrix theory, we can construct Feynman diagrams by e.g.,

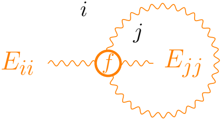


The Feynman rules are derived: an edge bordering i and j adds a factor $\frac{1}{f'[\lambda_i,\lambda_j]} = \frac{\lambda_i-\lambda_j}{f'(\lambda_i)-f'(\lambda_j)}$. Same Feynman rules as in [Belliard–Charbonnier–Eynard–Garcia-Failde,'21]!
As such we can define all diagrams with noncommutative vertices of arbitrary valence, their non-locality modulated by f.

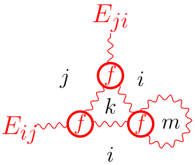
Examples:



$$= \sum_{k=1}^{N} \frac{f'[i,j,k,j]}{f'[j,k]}$$



$$= \frac{f'[i,j,j,i]}{f'[i,j]}$$



$$= \sum_{k,m=1}^{N} \frac{f'[j,i,k]f'[k,i,m,i]f'[j,k,i]}{f'[j,k]f'[i,k]^2}$$

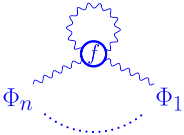
We let

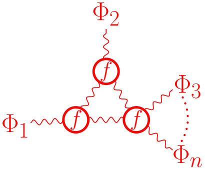
$$\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L}$$

be the sum of all relevant one-loop one-particle-irreducible n-point functions, whose external edges can naturally be labeled cyclically. The one-loop quantum effective spectral action is defined to be the formal series

$$\sum_{n=1}^{\infty} \frac{1}{n} \langle\!\langle \Phi, \ldots, \Phi \rangle\!\rangle_N^{1L}.$$

To note a few terms,

$$\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L} =$$ 

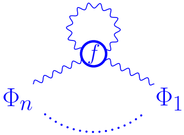
$$+$$ 

$$+ \ldots .$$

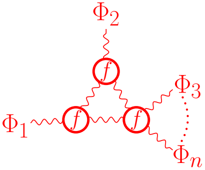
By definition, $\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L} = \langle\!\langle \Phi_2, \ldots, \Phi_n, \Phi_1 \rangle\!\rangle_N^{1L}$.

To note a few terms,

$$\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L} =$$



$$+$$



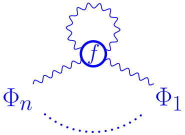
$$+ \ldots .$$

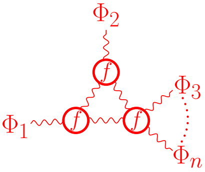
By definition, $\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L} = \langle\!\langle \Phi_2, \ldots, \Phi_n, \Phi_1 \rangle\!\rangle_N^{1L}$.
`Question:` does

$$\langle\!\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle\!\rangle_N^{1L} - \langle\!\langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle\!\rangle_N^{1L} = \langle\!\langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle\!\rangle_N^{1L}$$

hold as well?

To note a few terms,

$\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L} =$ 

$+$ 

$+ \ldots .$

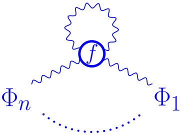
By definition, $\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L} = \langle\!\langle \Phi_2, \ldots, \Phi_n, \Phi_1 \rangle\!\rangle_N^{1L}$.
`Question:` does

$$\langle\!\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle\!\rangle_N^{1L} - \langle\!\langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle\!\rangle_N^{1L} = \langle\!\langle \Phi_1, [D, a], \Phi_2, \ldots, \Phi_n \rangle\!\rangle_N^{1L}$$

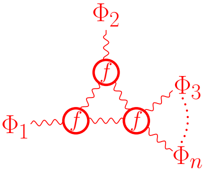
hold as well?
`Answer:` yes.

To note a few terms,

$$\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L} =$$



$+$



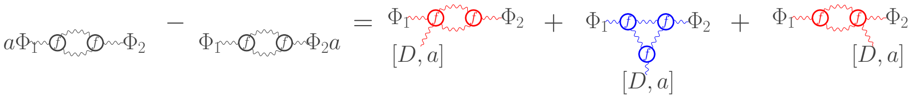
$+ \ldots .$

By definition, $\langle\!\langle \Phi_1, \ldots, \Phi_n \rangle\!\rangle_N^{1L} = \langle\!\langle \Phi_2, \ldots, \Phi_n, \Phi_1 \rangle\!\rangle_N^{1L}$.
`Question:` does

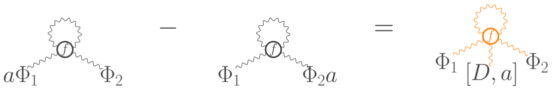
$$\langle\!\langle \Phi_1, a\Phi_2, \ldots, \Phi_n \rangle\!\rangle_N^{1L} - \langle\!\langle \Phi_1 a, \Phi_2, \ldots, \Phi_n a \rangle\!\rangle_N^{1L} = \langle\!\langle \Phi_1, [D,a], \Phi_2, \ldots, \Phi_n \rangle\!\rangle_N^{1L}$$

hold as well?
`Answer:` yes. cf. [vN–van Suijlekom, PoSiPM ’23]

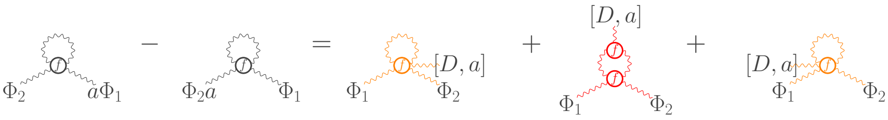
For example, the contributions to $\langle\!\langle a\Phi_1, \Phi_2\rangle\!\rangle_N^{1L} - \langle\!\langle \Phi_1, \Phi_2 a\rangle\!\rangle_N^{1L}$ are



and



and



We derive

$$\langle\!\langle a\Phi_1, \Phi_2\rangle\!\rangle_N^{1L} - \langle\!\langle \Phi_1, \Phi_2 a\rangle\!\rangle_N^{1L} = D, D\langle, a\langle, \Phi_1, \Phi_2\rangle\!\rangle_N^{1L}.$$

As the non-analytic part of our earlier theorem only depended on the cyclicity and commutation property of $\langle \cdots \rangle$, we conclude that the one-loop quantum effective spectral action takes the exact same form as the spectral action.

Theorem (vN–van Suijlekom)

There exist cyclic cocycles $(\psi_1^N, \psi_3^N, \ldots)$ and $(\phi_2^N, \phi_4^N, \ldots)$ such that for all 'finite-dimensional' $\Phi = \pi_D(A) \in \Omega_D^1(\mathcal{A})_{\text{s.a.}}$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \langle\langle \Phi, \ldots, \Phi \rangle\rangle_N^{1L} \sim \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}^N} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}^N} F(A)^k \right).$$

We can therefore absorb all one-loop divergences into the cyclic cocycles!

Open questions:

1. Can we do the same for higher loop?
2. Can we describe the renormalisation group flow of these cyclic cocycles?
3. Can we replace H_N by a subspace $\Omega^1_D(\mathcal{A})_{sa}$ (modulo gauge transformations)?
4. Can we treat the non-compact case (using multiple operator integration?)?
5. How to understand the difference between f Schwartz and f polynomial? Is there still a relation to TR in the more general case?

Help welcome!

Open questions:

1. Can we do the same for higher loop?
2. Can we describe the renormalisation group flow of these cyclic cocycles?
3. Can we replace H_N by a subspace $\Omega^1_D(\mathcal{A})_{sa}$ (modulo gauge transformations)?
4. Can we treat the non-compact case (using multiple operator integration?)?
5. How to understand the difference between f Schwartz and f polynomial? Is there still a relation to TR in the more general case?

Help welcome!

Thanks for attention :)