Cyclic Cocycles in the Spectral Action and One-Loop Corrections

Teun van Nuland

In celebration of Walter's appointment to
'hoogleraar Niet-commutatieve meetkunde'

Part 0:
Introduction

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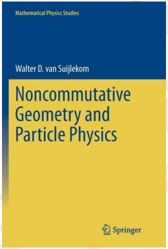
Physical effective actions can be obtained from this expansion, including and going beyond the Standard Model.
One uses usual RG flow to obtain physics at lower energy scales (lower Λ). This gives up a little bit of the elegance of the spectral action, admittedly.



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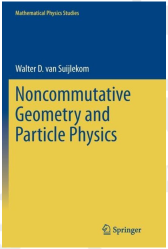

Connes–Chamseddine 2006:

$$\zeta_{D+A}(0) - \zeta_D(0) = \frac{1}{4}\int_{\tau_0}(dA + A^2)^2 - \frac{1}{2}\int_{\psi}(AdA + \frac{2}{3}A^3)$$

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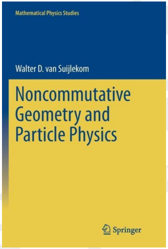

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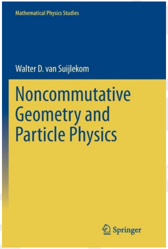

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   E.g. Taylor: $\mathrm{Tr}(f(D+A)) \sim \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{Tr}(\frac{d^n}{dt^n} f(D+tA)\big|_{t=0})$

Contents

Part 1:

Expansion of the Spectral Action

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Our goal is to expand

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for $V = \sum a_j[D, b_j] \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$ and obtain forms that depend on the universal 1-form $A = \sum a_j db_j \in \Omega^1(\mathcal{A})$.

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$$\Omega^1_D(\mathcal{A}) := \left\{\sum a_j[D, b_j] : \ a_j, b_j \in \mathcal{A}\right\} \subseteq \mathcal{B}(\mathcal{H}).$$

An important universal form is the field strength or curvature $F \in \Omega^2(\mathcal{A})$ of $A \in \Omega^1(\mathcal{A})$ given by

$$F := A^2 + dA.$$

Example: If $A = adb$, then $F = adbadb + dadb$ for $a, b \in \mathcal{A}$.

Another important universal form is the Chern–Simons form

$$\mathrm{cs}_{2k+1}(A) := \int_0^1 A(t^2A^2 + tdA)^k \, dt\,.$$

Example: $\mathrm{cs}_1(A) = A$, $\mathrm{cs}_3(A) = \frac{1}{3}A^3 + \frac{1}{2}AdA$, etc.

For suitable $f : \mathbb{R} \to \mathbb{C}$, we have the Taylor expansion

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We have that [Walter, JFA, 2010]

$$\frac{d^n}{dt^n} \mathrm{Tr}(f(D+tV))\big|_{t=0} = \frac{n!}{2\pi i} \oint_\gamma f'(z)\, \mathrm{Tr}(V(z-D)^{-1} \cdots V(z-D)^{-1}).$$

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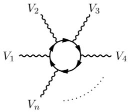

$$\langle V_1, \ldots, V_n \rangle := \frac{1}{2\pi i} \oint_\gamma f'(z)\, \mathrm{Tr}(V_1(z-D)^{-1} \cdots V_n(z-D)^{-1}),$$

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$$\int_{\phi_n} a_0 da_1 \cdots da_n := \phi_n(a^0,\ldots,a^n) := \langle a^0[D,a^1],[D,a^2],\ldots,[D,a^n]\rangle\,.$$

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For odd n, we get $b\phi_n = \phi_{n+1}$. Define

$$\psi_n := c_n(\phi_n - \tfrac{1}{2}B_0\phi_{n+1}).$$

Taking $c_{2k+1} := \frac{(-1)^k k!}{(2k+1)!}$, also $(\psi_1,\psi_3,\dots)$ forms a cyclic cocycle, i.e., a (b,B)-cocycle.

The only properties of $\langle ., \ldots, . \rangle$ that we need are

1. $\langle V_1, \ldots, V_n \rangle = \langle V_n, V_1, \ldots, V_{n-1} \rangle$.
2. $\langle aV_1, \ldots, V_n \rangle - \langle V_1, \ldots, V_n a \rangle = \langle V_1, \ldots, V_n, [D, a] \rangle$,

From this, one obtains $B\phi_{2k+2} = b\phi_{2k} = 0$ and $b\psi_{2k-1} + B\psi_{2k+1} = 0$.

Using just the rule

$$\langle V_1, \dots, V_j, aV_{j+1}, \dots, V_n \rangle - \langle V_1, \dots, V_j a, V_{j+1}, \dots, V_n \rangle$$
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$$\langle a[D, b], a[D, b], a[D, b]\rangle_3 = \int_{\phi_3} A^3 + \int_{\phi_4} AdAA + \int_{\phi_5} AdAdA$$

etcetera.

Define the function class

$$\mathcal{E}_s(\mathbb{R}) := \left\{ f \in C^\infty \;\middle|\; \exists_C \forall_{m \leqslant s} \forall_{n \geqslant 0} : \|\widehat{(fu^m)^{(n)}}\| \leqslant C^{n+1}\sqrt{n!} \right\},$$

where $u(x) = x - i$.

Theorem (vN–van Suijlekom)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an s-summable spectral triple and $f \in \mathcal{E}_s(\mathbb{R})$. Then there exist entire cyclic cocycles $(\phi_2, \phi_4, \ldots)$ and $(\psi_1, \psi_3, \ldots)$ such that for all $V = \pi_D(A) \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$:

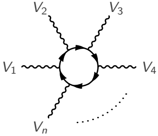
$$\mathrm{Tr}(f(D + tV) - f(D)) = \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right),$$

where $c_k := \frac{(2k-1)!}{(-1)^{k-1}(k-1)!}$. This series converges absolutely.

Part 2:

One-Loop corrections to the Spectral Action

We introduce the notation

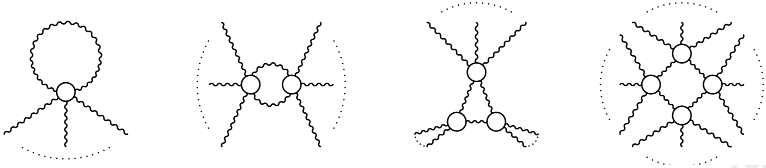


$$:= \quad \langle V_1, \ldots, V_n \rangle.$$

This represents a noncommutative vertex, and the sum of which represents a classical action (the spectral action). This makes sense, because the above brackets are analogues of functional derivatives. We would like to quantum-correct this classical action, but without giving up on the paradigm of spectral triples.

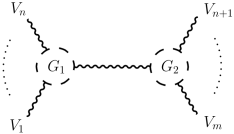
We wish to incorporate more Feynman diagrams, like

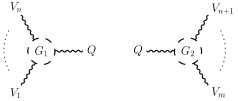
and (with clockwise NC vertices)



We will stay at one-loop for now.

By employing random matrix theory:





$$:= -\frac{\int_{H_N} \left(\cdots \right) e^{-\frac{1}{2}\langle Q,Q\rangle} dQ}{\int_{H_N} e^{-\frac{1}{2}\langle Q,Q\rangle} dQ}.$$

We let

$$\langle\!\langle V_1, \ldots, V_n \rangle\!\rangle_N^{1L}$$

be the sum of all relevant one-loop one-particle-irreducible n-point functions, whose external edges can naturally be labeled cyclically. one-loop quantum effective spectral action is defined to be the formal series

$$\sum_{n=1}^{\infty} \frac{1}{n} \langle\!\langle V, \ldots, V \rangle\!\rangle_N^{1L}.$$

The one-loop quantum effective spectral action surprisingly takes the exact same form as the spectral action.

Theorem (vN–van Suijlekom)

There exist cyclic cocycles $(\psi_1^N, \psi_3^N, \dots)$ and $(\phi_2^N, \phi_4^N, \dots)$ such that for all $V = \pi_D(A) \in \Omega_D^1(\mathcal{A})_{\text{s.a.}}$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \langle\langle V, \dots, V \rangle\rangle_N^{1L} \sim \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}^N} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}^N} F^k \right).$$

We thus obtain ‘corrective’ cyclic cocycles, indicating a renormalization flow taking place.

Although this is the end of this presentation, I hope it is just the start of a fruitful program. Walter, congratulations with your full professorship, and I look forward to our future collaborations!