How to Quantize an Observable Algebra

In Three ~~Easy~~ Steps

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Step 1: Choose a commutative C*-algebra

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...why?

Why a C*-algebra?

Let's take a look at canonical quantization. Beware that

$$[\hat{x}, \hat{p}] = i\hbar$$

doesn't determine $\hat{x}$ and $\hat{p}$ uniquely. Better:

$$e^{it_1\hat{x}}e^{it_2\hat{p}} = e^{-it_1 t_2\hbar}e^{it_2\hat{p}}e^{it_1\hat{x}}$$

Remember Stone's Theorem:

$$e^{it\hat{A}}\ (t \in \mathbb{R}) \quad \longleftrightarrow \quad \hat{A}$$

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Replace polynomials in $\hat{p}, \hat{x}$ by $\mathcal{W}(\mathbb{R}^2) := C^*\left(e^{i(t_1\hat{p}+t_2\hat{x})} \;\middle|\; t \in \mathbb{R}^2\right)$.

Weyl CCR Algebra

Let (X, σ) be symplectic such that $\overline{X}$ is a Hilbert space. On bosonic Fock space $\mathcal{F}_b(\overline{X})$, define the field

$$\phi(x) := \sqrt{\hbar}(a(x) + a(x)^\dagger)\,,$$

where $a(x)$ is annihilation of $x \in X$. This ϕ is real-linear and satisfies

$$e^{i\phi(x)}e^{i\phi(y)} = e^{-i\hbar\sigma(x,y)}e^{i\phi(y)}e^{i\phi(x)}\,.$$

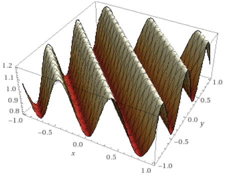
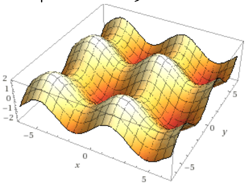
The Weyl CCR algebra $\mathcal{W}(X, \sigma)$ is the C*-subalgebra of $B(\mathcal{F}_b(X))$ generated by $e^{i\phi(x)}$.

Step 1: Choose a commutative C*-algebra

Step 1: Choose a *commutative* C*-algebra

If you know the quantum algebra, you can make an educated guess.

Binz, Honegger, Rieckers:
$\mathcal{W}^0(X) := \overline{\operatorname{span}}\left\{ e^{ix\cdot} \;\middle|\; x \in X \right\}$

Step 2: Define a quantization map

Weyl quantization
In finite dimensions: $Q_\hbar^W(f) := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \tilde{f}(x) e^{i(x_1 \cdot \hat{x} + x_2 \cdot \hat{p})}$
For the Weyl algebra: $Q_\hbar^W(e^{ix\cdot}) := e^{i\phi(x)}$ and linearly extended

Berezin quantization, Wick quantization, etc.

Step 2: Define a quantization map *on a dense Poisson* $\tilde{A}_0 \subseteq A_0$

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For the Weyl algebra: $Q_\hbar^W(e^{ix\cdot}) := e^{i\phi(x)}$ and linearly extended to $\text{span}\{e^{ix\cdot}\}$.

Berezin quantization, Wick quantization, etc.

Step 3: Check whether $Q_\hbar : \tilde{A}_0 \to A_\hbar$ is a strict deformation quantization.

Definition

*Let $\tilde{A}_0$ be a complex Poisson *-algebra, i.e.: $\{f,g\}^* = \{f^*, g^*\}$.*
A **strict deformation quantization** *of $\tilde{A}_0$ over $I = [0,1]$ is a collection of C*-algebras $\{A_\hbar\}_{\hbar \in I}$, with injective linear *-preserving maps*

$$\{Q_\hbar : \tilde{A}_0 \to A_\hbar\}_{\hbar \in I}\,,$$

*such that Q_0 is the identity, $Q_\hbar(\tilde{A}_0)$ is a dense *-subalgebra of $A_\hbar$, and for all $f, g \in \tilde{A}_0$:*

$$\hbar \mapsto \|Q_\hbar(f)\|_\hbar \text{ is continuous on } I, \tag{I}$$

$$\lim_{\hbar \to 0} \|Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)\|_\hbar = 0\,, \tag{II}$$

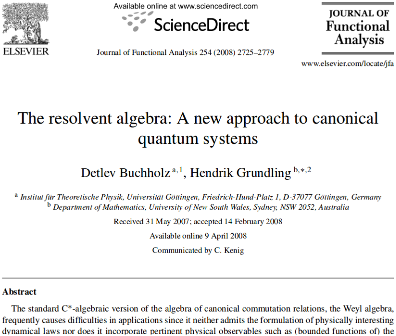
$$\lim_{\hbar \to 0} \left\| \tfrac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f,g\}) \right\|_\hbar = 0\,. \tag{III}$$

Resolvent Algebra:

$$\mathcal{R}(X,\sigma) := C^*(R(\lambda,x) \mid \lambda \in \mathbb{R}\backslash 0, x \in X)$$

where $R(\lambda,x) := (i\lambda - \phi(x))^{-1}$ are the resolvents of $\phi(x)$.

(Buchholz–Grundling, 2008)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The resolvent algebra: A new approach to canonical quantum systems

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Abstract

The standard C*-algebraic version of the algebra of canonical commutation relations, the Weyl algebra, frequently causes difficulties in applications since it neither admits the formulation of physically interesting dynamical laws nor does it incorporate pertinent physical observables such as (bounded functions of) the

The Resolvent Algebra

$$\mathcal{R}(X,\sigma) := C^*((i\lambda - \phi(x))^{-1} \mid \lambda \in \mathbb{R}\backslash 0, x \in X)$$

Features:

- Conjugating with e^{itH} preserves $\mathcal{R}(X,\sigma)$ for many H, like $H = \hat{p}^2 + V(\hat{x})$ for any $V \in C_0(\mathbb{R})$.
- $\mathcal{R}(X,\sigma)$ has ideals, but every regular representation is faithful.
- $\mathcal{R}(\mathbb{R}^{2n},\sigma_n)$ contains the compacts, and $\mathcal{R}$ behaves nice w.r.t. direct limits.

(Step 1)

Define the **commutative resolvent algebra** as

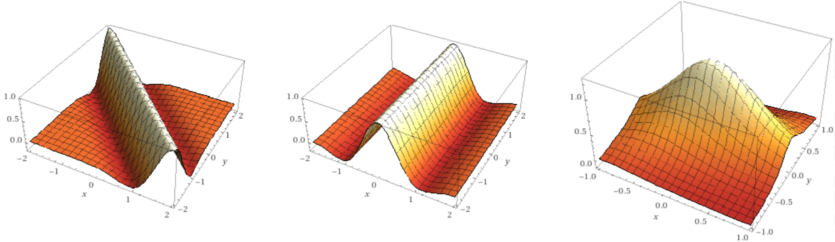
$$C_{\mathcal{R}}(X) := C^* \left(h_x^\lambda \;\middle|\; \lambda \in \mathbb{R}\backslash 0,\ x \in X \right)$$

where $h_x^\lambda(y) := \frac{1}{i\lambda - x\cdot y}$

(recall $R(\lambda, x) := (i\lambda - \phi(x))^{-1}$).

(Step 1)

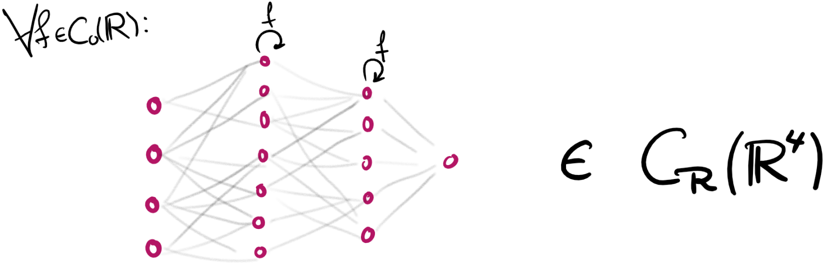
$$\mathcal{S}_{\mathcal{R}}(X) := \text{span}\,\{g \circ P_V \mid V \subseteq X \text{ fin.dim}, \; g \in \mathcal{S}(V)\}$$



$$(g_1 \circ P_V) \quad \cdot \quad (g_2 \circ P_W) \quad = \quad g \circ P_{V+W}$$

Lemma

*$\mathcal{S}_{\mathcal{R}}(X)$ is a dense *-subalgebra of $C_{\mathcal{R}}(X)$*



So:

$C_{\mathcal{R}}(\mathbb{R}^n)$ is in many ways a natural algebra.

(Step 2)

Suppose $f \in \mathcal{S}(\mathbb{R}^2)$, write

$$f(y) = \int_{\mathbb{R}^2} đx\, \tilde{f}(x) e^{ix\cdot y} \qquad (đx = \frac{dx}{(2\pi)})$$

Define

$$Q_\hbar^W(f) := \int_{\mathbb{R}^2} đx\, \tilde{f}(x) e^{i(x_1\hat{x}+x_2\hat{p})}$$

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$\mathbb{R}^{2n} \to X$, problem: $\mathcal{S}(X) = \{0\}$.
Solution: $\mathcal{S}_\mathcal{R}(X) := \text{span}\,\{g \circ P_V \mid V \subseteq X \text{ fin.dim}, \, g \in \mathcal{S}(V)\}$

$$Q_\hbar^W(g \circ P_V) := \int_V đx\, \tilde{g}(x) e^{i\phi(x)}$$

(<u>Step 3</u>)

Now $Q_\hbar^W(h_x^\lambda) = \int_{\mathbb{R}} đt \, (\frac{1}{i\lambda - \cdot})\tilde{}(t) e^{it\phi(x)} = R(\lambda, x)$.

Theorem

$Q_\hbar^W : \mathcal{S}_{\mathcal{R}}(X) \to \mathcal{R}(X, \sigma)$ *is a strict deformation quantization.*

Bundles

(Joint work with R. Stienstra)
(Part of a bigger project with also F. Arici and W. van Suijlekom)

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Let us quantize a principal G-bundle over M,

where M is finite and $G = U(1)$.
I.e., quantize $G^n = \mathbb{T}^n$ *with* $n \to \infty$ *in mind*

(Step 1)

Take $X = T^*\mathbb{T}^n$. Define
$C_{\mathcal{R}}(X) := C^*(g \circ \pi_x \mid x \in \mathbb{R}^{2n},\ g \in C_0(X/\{x\}^\perp))$
$\mathcal{S}_{\mathcal{R}}(X) := \mathrm{span}\{e^{ix\cdot} g \circ P_V\} \otimes C^\infty(\mathbb{T}^n)$

(Step 2)

$$Q_\hbar^W(f) := \int_{\mathbb{R}^n} đq \int_{\mathbb{R}^n} đp\ \tilde{f}(q,p) U_{q,p}$$

where

$$U_{q,p}\psi(x) := e^{iq(x+p\hbar/2)}\psi(x + \hbar p)$$

(Step 3-ϵ)

Theorem

Except maybe for (I, Rieffel), $Q_\hbar^W : \mathcal{S}_{\mathcal{R}}(X) \to C_{\mathcal{R}}(X)$ is a strict quantization.

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A very positive bycatch: Both $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ and $\mathcal{R}(T^*\mathbb{T}^n) := C^*(\mathrm{im} Q_\hbar^W)$ are conserved under time evolution!

Summary:

How to quantize an observable algebra?

Step 1: Choose a commutative C*-algebra
Step 2: Define a quantization map on a dense Poisson $\tilde{A}_0 \subseteq A_0$
Step 3: Check whether $Q_\hbar : \tilde{A}_0 \to A^\hbar$ is a strict deformation quantization.

This procedure can yield fruitful results!