Spotting Multiple Operator Integrals in nature

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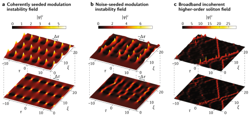
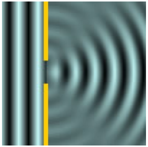
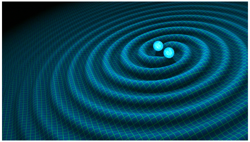
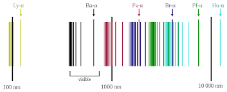
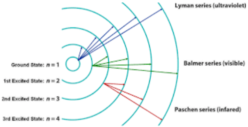
Nature likes to sit in the spectrum of a self-adjoint operator D.

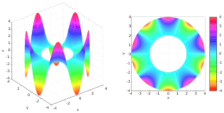


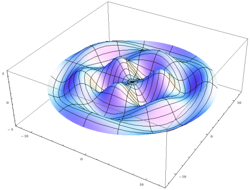












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We would like to understand variational properties of the spectral action as $D \mapsto D + V$, $V \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$. For instance, a Taylor expansion

$$\mathrm{Tr}(f(D+V)) = \mathrm{Tr}(f(D)) + \sum_{n=1}^{\infty} \frac{1}{n!} \mathrm{Tr}\left(\frac{d^n}{dt^n}\bigg|_{t=0} f(D+tV) \right).$$

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Examples:

1. Quantum Mechanics ($D = H$ Hamiltonian, V potential, $f(t) = e^{it}$ or $f(t) = \frac{1}{i-t}$)
2. Noncommutative geometry (D Dirac operator, V gauge field)
3. Noncommutative QFT (D Dirac operator, V quantum field)
4. Spectral geometry, heat expansion ($D = \Delta$ Laplacian, $f(t) = e^{-t}$)

Let $\mathcal{H}$ be a Hilbert space, D a (possibly unbounded) self-adjoint operator in $\mathcal{H}$, $V^* = V \in \mathcal{B}(\mathcal{H})$. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and of rapid decrease, which acts by functional calculus on D or $D + V$. We want to know

$$\frac{d^n}{dt^n} f(D + tV)|_{t=0}.$$

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For $f(t) = \frac{1}{z-t}$ $(z \in \mathbb{C} \setminus \mathbb{R})$ this becomes easy. Then

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Now for any f. Trick! Write

$$f(D+tV) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-D-tV)^{-1}\,dz.$$

Recall $\frac{d}{dt}\Big|_{t=0}(z-D-tV)^{-1} = (z-D)^{-1}V(z-D)^{-1}$. We obtain

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Example: $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In our example,

$$\frac{d}{dt}\Big|_{t=0} f(D+tV) = \frac{1}{2\pi i}\int_\Gamma (z-D)^{-1}V(z-D)^{-1}\,dz$$

$$= \frac{1}{2\pi i}\int_\Gamma \begin{pmatrix} (z-\lambda_1)^{-1} & 0 \\ 0 & (z-\lambda_2)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (z-\lambda_1)^{-1} & 0 \\ 0 & (z-\lambda_2)^{-1} \end{pmatrix} dz$$

$$= \begin{pmatrix} 0 & \frac{f(\lambda_1)-f(\lambda_2)}{\lambda_1-\lambda_2} \\ \frac{f(\lambda_1)-f(\lambda_2)}{\lambda_1-\lambda_2} & 0 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 0 & f^{[1]}(\lambda_1,\lambda_2) \\ f^{[1]}(\lambda_1,\lambda_2) & 0 \end{pmatrix}$$

where we have introduced the divided difference $f^{[1]}(\lambda_0,\lambda_1) = \frac{f(\lambda_0)-f(\lambda_1)}{\lambda_0-\lambda_1}$.
More generally, $f^{[n]}(\lambda_0,\ldots,\lambda_n) = \frac{f^{[n-1]}(\lambda_0,\ldots,\lambda_{n-1})-f(\lambda_1,\ldots,\lambda_n)}{\lambda_0-\lambda_n}$.

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How to understand $\frac{1}{n!}\frac{d^n}{dt^n}\Big|_{t=0} f(D+tV)$?

$$\frac{1}{2\pi i}\int_\Gamma f(z)(z-D)^{-1}V_1(z-D)^{-1}\cdots V_n(z-D)^{-1}\,dz$$

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Suppose $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is measurable and can be written as

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Sigma d\sigma\, \alpha_0(\lambda_0, \sigma) \cdots \alpha_n(\lambda_n, \sigma)$$

for some measure space (Σ, σ), and each $\alpha_j : \Sigma \times \mathbb{R} \to \mathbb{R}$ is nice enough. We define [Peller 2006, Azamov, Carey, Dodds, Sukochev, 2009] the Multiple Operator Integral

$$T_\phi^D(V_1, \ldots, V_n) := \int_\Sigma d\sigma\, \alpha_0(D, \sigma) V_1 \alpha_1(D, \sigma) \cdots V_n \alpha_n(D, \sigma).$$

This defines a (well-defined!) multilinear operator

$$T_\phi^D : \mathcal{B}(\mathcal{H}) \times \cdots \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\,.$$

One often sees $\phi = f^{[n]}$, not in the least because

$$\frac{1}{n!} \frac{d^n}{dt^n} f(D + tV) = T_{f^{[n]}}^D(V, \ldots, V).$$

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Credit $n = 1$: Daletskii, Krein, Löwner, Krein, Birman, Solomyak

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Résumé: The Multiple Operator Integral is a multilinear operator

$$T_\phi^D : \mathcal{B}(\mathcal{H}) \times \cdots \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) ,$$

defined by

$$T_\phi^D(V_1, \ldots, V_n) := \int_\Sigma d\sigma \, \alpha_0(D,\sigma) V_1 \alpha_1(D,\sigma) \cdots V_n \alpha_n(D,\sigma).$$

When $V_1, \ldots, V_n \in \mathcal{S}^n$ then $T_\phi^D(V_1, \ldots, V_n) \in \mathcal{S}^1$.
However, V is often not in $\mathcal{S}^n$. (E.g., multiplication operator.) Sometimes $(D - i)^{-1}$ is, and more often $V(D-i)^{-1}$ is.

Suppose that D is selfadjoint in a Hilbert space $\mathcal{H}$, and $V = V^* \in \mathcal{B}(\mathcal{H})$, such that $V(D-i)^{-1} \in \mathcal{S}^n$ is Schatten n-class. Suppose $f \in C^1$ s.t. $\hat{f}, \widehat{f'}, \widehat{(fu)}' \in L^1$. Let $u(x) := x - i$. We have

$$(fu)^{[1]}(\lambda_0, \lambda_1) = f(\lambda_0, \lambda_1) + f^{[1]}(\lambda_0, \lambda_1)u(\lambda_1)$$

so

$$f^{[1]}(\lambda_0, \lambda_1) = (fu)^{[1]}(\lambda_0, \lambda_1)(\lambda_1 - i)^{-1} - f(\lambda_0)(\lambda_1 - i)^{-1}.$$

Therefore,

$$T^D_{f^{[1]}}(V) = T^D_{(fu)^{[1]}}(V(D-i)^{-1}) - f(D)V(D-i)^{-1}.$$

Hence $T^D_{f^{[1]}}(V) \in \mathcal{S}^n$. [Skripka, Chattopadhyay, 2013]

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Hence $T^D_{f^{[1]}}(V) \in \mathcal{S}^n$. [Skripka, Chattopadhyay, 2013] This argument extends to higher order and to more relaxed assumptions on V and D. In particular,

$$V_j(D-i)^{-1} \in \mathcal{S}^n \Rightarrow T^D_{f^{[n]}}(V_1, \ldots, V_n) \in \mathcal{S}^1.$$

[Skripka, vN, 2021]

Applications in nature:

1. Spectral shift function for operators in Quantum Mechanics
2. Cyclic cocycles in noncommutative geometry
3. Noncommutative quantum field theory
4. Heat-trace expansion

Application 1: we find function η_n [Skripka, vN, 2021] such that

$$\mathrm{Tr}(f(D+V)) - \sum_{k=0}^{n-1} \frac{d^n}{dt^n} \mathrm{Tr}(f(D+V))\Big|_{t=0} = \int_{\mathbb{R}} \eta_n(t) f^{(n)}(t)\, dt\,,$$

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Our milder assumptions bridge the gap between theory and application.
Examples are $D = \Delta$, $D = \Delta + V$, $D = \mathcal{D}$ in quantum theory and NCG.

Application 2:

Theorem (van Suijlekom, vN, 2021)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an s-summable spectral triple.

$$\mathrm{Tr}(f(D+V) - f(D)) = \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right),$$

where $c_k := \frac{(2k-1)!}{(-1)^{k-1}(k-1)!}$. This series converges absolutely.

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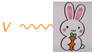
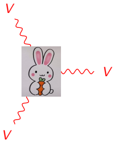
$$\mathrm{Tr}(f(D+V) - f(D)) = \sum_{n=1}^{\infty} \mathrm{Tr}(T^D_{f^{[n]}}(V, \dots, V))$$

$=$



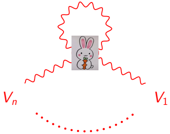
$+ \frac{1}{2}$

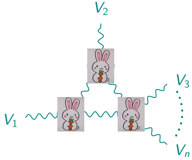
$+ \frac{1}{3}$



$+ \dots .$

Application 3: Introducing quantum fields,

$$\langle\!\langle V_1, \ldots, V_n \rangle\!\rangle_N^{1L} =$$ 

$+$ 

$+ \ldots,$

we obtain

Theorem (van Suijlekom, vN, 2022)

There exist cyclic cocycles $(\psi_1^N, \psi_3^N, \ldots)$ and $(\phi_2^N, \phi_4^N, \ldots)$ such that for all 'finite-dimensional' $V = \sum_j a_j[D, b_j]$, $A = \sum_j a_j db_j$, we have asymptotically

$$\sum_{n=1}^{\infty} \frac{1}{n} \langle\!\langle V, \ldots, V \rangle\!\rangle_N^{1L} \sim \sum_{k=1}^{\infty} \left(c_k \int_{\psi_{2k-1}^N} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}^N} F^k \right).$$

Application 4: Local invariants in the asymptotics of

$$\mathrm{Tr}(e^{-t\Delta})$$

as $t \to 0$ for the Laplacian Δ of any conformally deformed noncommutative torus. Previously only in special cases.
[Connes,Moscovici,Tretkoff,Fathizadeh,Khalkhali,many more] Answer in higher order and for any dimension: uses multiple operator integrals.
[Sukochev, Zanin, vN, 2023]

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Suppose you can calculate or estimate $[D, V]$. What is $[f(D), V]$ for a function f?

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Suppose you can calculate or estimate $[D, V]$. What is $[f(D), V]$ for a function f?
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Thank you for your attention