Strict Deformation Quantization in Infinite Dimensions

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Part 1:
Introduction

Deformation Quantization: mathematical approach to quantization in physics.

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2. Strict: $\hbar \in \mathbb{R}$

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a Poisson subalgebra: $\mathcal{A}_0 \subseteq A_0$;
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For example:

Write $f \in \mathcal{S}(\mathbb{R}^{2n}) \subseteq C_0(\mathbb{R}^{2n})$ as

$$f(x,p) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \tilde{f}(y) e^{i(y_1 \cdot x + y_2 \cdot p)}\, dy$$

and define

$$Q_\hbar^W(f) := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \tilde{f}(y) e^{i(y_1 \cdot \hat{x} + y_2 \cdot \hat{p})}\, dy \in K(L^2(\mathbb{R}^n)).$$

Canonical Quantization

Canonical quantization usually starts by postulating commutation relations. However, beware that

$$[\hat{x}, \hat{p}] = i\hbar$$

doesn't determine $\hat{x}$ and $\hat{p}$ uniquely. Better:

$$e^{it_1\hat{x}}e^{it_2\hat{p}} = e^{-it_1t_2\hbar}e^{it_2\hat{p}}e^{it_1\hat{x}}$$

Remember Stone's Theorem:

$$e^{it\hat{a}}\ (t \in \mathbb{R}) \quad \longleftrightarrow \quad \hat{a}$$

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Replace polynomials in $\hat{p}, \hat{x}$ by $W(\mathbb{R}^2) := C^* \left(e^{i(t_1\hat{p} + t_2\hat{x})} \;\middle|\; t \in \mathbb{R}^2 \right)$.

Part 2:

Infinite Dimensional Quantum Systems

Weyl C*-algebra (Weyl CCR algebra)

Let (X, σ) be symplectic such that $\overline{X}$ is a Hilbert space. On bosonic Fock space $\mathcal{F}_b(\overline{X})$, define the field

$$\phi(x) := \sqrt{\hbar}(a(x) + a(x)^\dagger),$$

where $a(x)$ is annihilation of $x \in X$. This ϕ is real-linear and satisfies

$$e^{i\phi(x)} e^{i\phi(y)} = e^{-i\hbar\sigma(x,y)} e^{i\phi(y)} e^{i\phi(x)}.$$

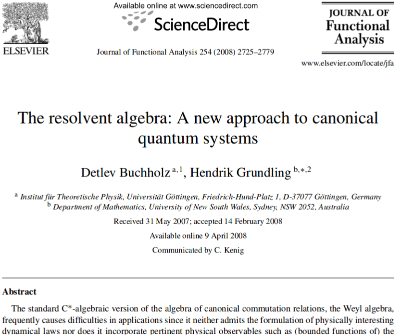
The Weyl C*-algebra $W(X, \sigma)$ is the C*-subalgebra of $B(\mathcal{F}_b(X))$ generated by $e^{i\phi(x)}$.

Resolvent Algebra:

$$\mathcal{R}(X,\sigma) := C^*(R(\lambda,x) \mid \lambda \in \mathbb{R}\backslash 0, x \in X)$$

where $R(\lambda,x) := (i\lambda - \phi(x))^{-1}$ are the resolvents of $\phi(x)$.

(Buchholz–Grundling, 2008)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The resolvent algebra: A new approach to canonical quantum systems

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Abstract

The standard C*-algebraic version of the algebra of canonical commutation relations, the Weyl algebra, frequently causes difficulties in applications since it neither admits the formulation of physically interesting dynamical laws nor does it incorporate pertinent physical observables such as (bounded functions of) the

$$\mathcal{R}(X,\sigma) := C^*((i\lambda - \phi(x))^{-1} \mid \lambda \in \mathbb{R}\backslash 0, x \in X)$$

Features:

- $\mathcal{R}(X,\sigma)$ has ideals, but every regular representation is faithful.
- $\mathcal{R}(\mathbb{R}^{2n},\sigma_n)$ contains the compacts, and $\mathcal{R}$ behaves nice w.r.t. direct limits.
- Conjugating with e^{itH} preserves $\mathcal{R}(X,\sigma)$ for many H, like $H = \hat{p}^2 + V(\hat{x})$ for any $V \in C_0(\mathbb{R})$.

Part 3:
Strict Deformation Quantization

Definition

Let $\mathcal{A}_0$ *be a complex Poisson *-algebra, i.e.:* $\{f,g\}^* = \{f^*, g^*\}$. *A* **strict deformation quantization** *of* $\mathcal{A}_0$ *over* $I = [0,1]$ *is a collection of C*-algebras* $\{A_\hbar\}_{\hbar\in I}$, *with injective linear *-preserving maps*

$$\{Q_\hbar : \mathcal{A}_0 \to A_\hbar\}_{\hbar\in I}\,,$$

such that Q_0 *is the identity,* $Q_\hbar(\mathcal{A}_0)$ *is a dense *-subalgebra of* $A_\hbar$, *and for all* $f,g\in\mathcal{A}_0$*:*

$$\hbar \mapsto \|Q_\hbar(f)\|_\hbar \text{ is continuous on } I\,, \tag{I}$$

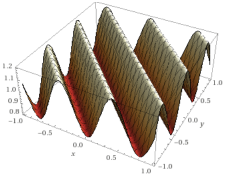
$$\lim_{\hbar\to 0} \|Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)\|_\hbar = 0\,, \tag{II}$$

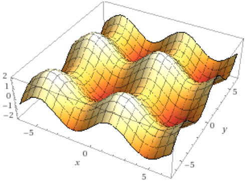
$$\lim_{\hbar\to 0} \left\|\frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f,g\})\right\|_\hbar = 0\,. \tag{III}$$

Weyl quantization of the Weyl algebra

In this case $A_\hbar = W(X, \sigma)$. We can make an educated guess for the commutative C*-algebra A_0.

Binz, Honegger, Rieckers, 2004:
$A_0 = W_0(X) := \overline{\text{span}} \left\{ e^{ix\cdot} \;\middle|\; x \in X \right\}$





For the dense *-subalgebra $\mathcal{A}_0 \subseteq A_0$ with Poisson structure we take

$$\mathcal{W}_0(X) := \text{span}\{e^{ix\cdot}\}.$$

Weyl quantization:
Recall, in finite dimensions:

$$Q_\hbar^W(f) := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \tilde{f}(y) e^{i(y_1\cdot\hat{x}+y_2\cdot\hat{p})}\, dy$$

For the Weyl algebra:

$$Q_\hbar^W(e^{ix\cdot}) := e^{i\phi(x)}$$

and linearly extended to $\mathcal{W}_0(X) := \text{span}\{e^{ix\cdot}\}$.

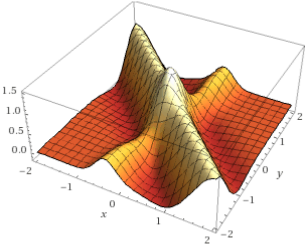
Quantization of the Resolvent Algebra

Define the **commutative resolvent algebra** as

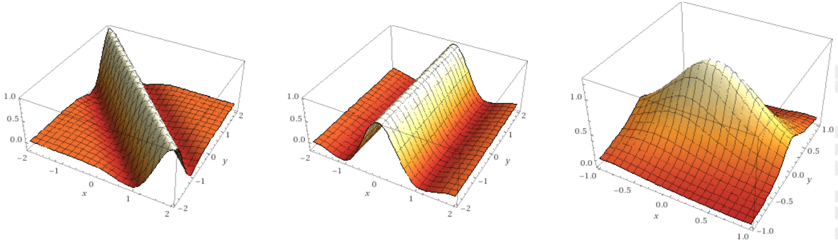
$$C_{\mathcal{R}}(X) := C^* \left(h_x^\lambda \;\middle|\; \lambda \in \mathbb{R} \setminus 0,\ x \in X \right)$$

where $h_x^\lambda(y) := \frac{1}{i\lambda - x \cdot y}$

(recall $R(\lambda, x) := (i\lambda - \phi(x))^{-1}$).



$$\mathcal{S}_{\mathcal{R}}(X) := \text{span}\,\{g \circ P_V \mid V \subseteq X \text{ fin.dim, } g \in \mathcal{S}(V)\}$$



$$(g_1 \circ P_V) \quad \cdot \quad (g_2 \circ P_W) \quad = \quad g \circ P_{V+W}$$

Lemma

*$\mathcal{S}_{\mathcal{R}}(X)$ is a dense *-subalgebra of $C_{\mathcal{R}}(X)$*

$\mathbb{R}^{2n} \to X$, problem: $\mathcal{S}(X) = \{0\}$.

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Now $Q_\hbar^W(h_x^\lambda) = \int_{\mathbb{R}} đt \left(\frac{1}{i\lambda - \cdot}\right)\tilde{}\,(t) e^{it\phi(x)} = R(\lambda, x)$.

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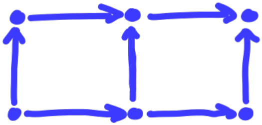
Theorem

$Q_\hbar^W : \mathcal{S}_{\mathcal{R}}(X) \to \mathcal{R}(X, \sigma)$ is a strict deformation quantization.

Part 4:
Lattice Gauge Theory

Lattice gauge theory gives us a selection of observables on the phase space of gauge fields. Fix t and choose a "lattice" ℓ in the time-slice $\mathbb{R}^3$.

Integration/parallel transport gives rise to a gauge group element $g \in G$ on each edge, i.e., the gauge field on this lattice is characterised by an element in G^ℓ (not yet quotienting out gauge invariance).

$\ell =$ 

Configuration space: G^ℓ
Phase space: $T^*(G^\ell) = T^*G^\ell$

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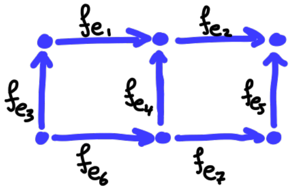
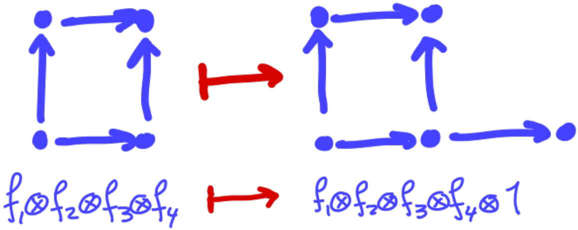


$f = \otimes_{e \in \ell} f_e : T^* G^\ell \to \mathbb{C}$,
and sums of such f

Observables are given by functions on the tangent bundle.

Classical *-algebras for every finite lattice



To eventually take the IR limit, one needs at least a unital algebra, like

$$\mathcal{W}_0(T^*G^\ell) := C^\infty(G^\ell) \otimes \mathcal{W}_0((\mathfrak{g}^*)^\ell).$$

Taking $G = U(1) = \mathbb{T}$

$$Q_\hbar^\ell : \mathcal{W}_0(T^*\mathbb{T}^\ell) \to \mathcal{W}_\hbar(T^*\mathbb{T}^\ell) \subseteq \mathcal{B}(L^2(\mathbb{T}^\ell)),$$
$$Q_\hbar^\ell(g \otimes e^{i\xi\cdot})\psi[x] := g[x + \tfrac{1}{2}\hbar\xi]\psi[x + \hbar\xi].$$

Theorem (Ruben Stienstra and T)

Except for injectivity and Rieffels condition,
$Q_\hbar^\ell : \mathcal{W}_0(T^*\mathbb{T}^\ell) \to \mathcal{W}_\hbar(T^*\mathbb{T}^\ell)$ *is a strict quantization.*

Injectivity and Rieffels condition (for positive $\hbar$) are false.

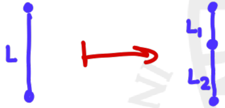
Even worse, there is no natural subdivision map on the quantum algebras.

There is hope. Define the operator systems

$$\tilde{\mathcal{W}}_0(T^*\mathbb{T}^\ell) := \overline{\text{span}}\{g \otimes e^{i\xi\cdot} : \ \|\xi_e\| \leqslant 1\},$$

restricted to which $Q_\hbar^\ell$ is injective.

Then we can define classical and quantum embedding maps for subdivisions of edges.
Namely,

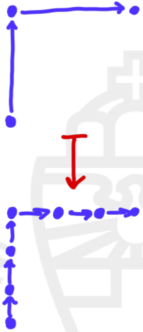


$$F_C^{\text{sub}} : g \otimes e^{i\xi\cdot} \mapsto (g \circ \mu) \otimes e^{i(\frac{L_1\xi}{L}, \frac{L_2\xi}{L})\cdot},$$

and

$$F_Q^{\text{sub}} : Q_\hbar^\ell(f) \mapsto Q_\hbar^m(F_C^{\text{sub}}(f)).$$

Suppose $f \in \mathcal{W}_0^\ell = \mathcal{W}_0(T^*\mathbb{T}^\ell)$,
for instance $f = g \otimes e^{i(\xi_1,\xi_2)\cdot}$.

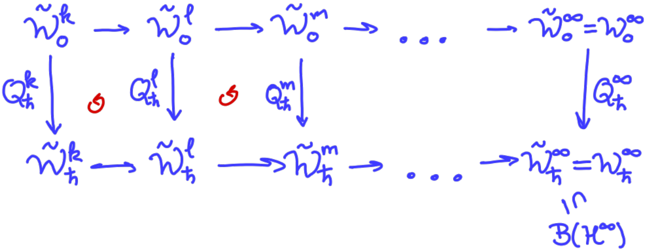


Then $F_C^{m\ell}(f) \in \tilde{\mathcal{W}}_0^m = \tilde{\mathcal{W}}_0(T^*\mathbb{T}^m)$ since, e.g.,
$F_C^{m\ell}(f) = (g \circ \mu^{\ell m}) \otimes e^{i(\frac{1}{3}\xi_1,\frac{1}{3}\xi_1,\frac{1}{3}\xi_1,\frac{1}{3}\xi_2,\frac{1}{3}\xi_2,\frac{1}{3}\xi_2)\cdot}$

In particular, the direct limit

$$\mathcal{W}_0^\infty := \lim_{\rightarrow} \tilde{\mathcal{W}}_0^\ell$$

is an algebra, and similarly for the quantum algebra.


$\tilde{W}_0^k \to \tilde{W}_0^l \to \tilde{W}_0^m \to \dots \to \tilde{W}_0^\infty = W_0^\infty$
$Q_\hbar^k$
$Q_\hbar^l$
$Q_\hbar^m$
$Q_\hbar^\infty$
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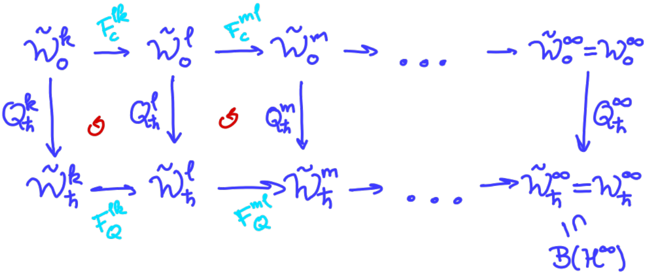
$\subset$
$B(\mathcal{H}^\infty)$


$\tilde{W}_0^k$
F_c^{lk}
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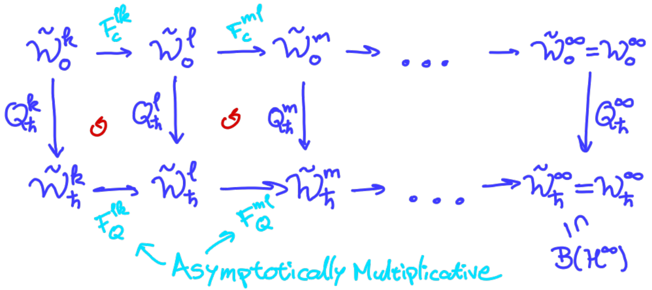

Theorem

$Q_\hbar^\infty : \mathcal{W}_0^\infty \to \mathcal{W}_\hbar^\infty$ is a strict deformation quantization. Moreover, $\mathcal{W}_0^\infty$ and $\mathcal{W}_\hbar^\infty$ are both closed under free time evolution.